EN 257: Applied Stochastic Processes
Problem Set 1

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Problem 1.50

You are a contestant on a game show. There are three closed doors leading to three rooms. Two of the rooms contain nothing, but the third contains a luxury automobile which is yours if you pick the right door. You are asked to pick a door by the MC who knows which room contains the car. After you pick a door, the MC opens a door (not the one you picked) to show an empty room. Show that, even without any further knowledge, you will greatly increase your chances of winning the car if you *switch* your choice from the door you originally picked.

This is a classic problem in probability and the "counter-intuitive" result can best be seen by applying Bayes' Theorem. To begin our analysis let's enumerable the sample space. Let O_i correspond to the event where the MC opens door i. In addition, let C_i be the event that the car is behind door i. Without loss of generality, we can assume that you always initially choose the first door and that the MC chooses the second (since we could always permute the door labels to achieve this condition). Subject to this condition, the sample space Ω can be enumerated simply by the position of the car as $\Omega = \{C_1, C_2, C_3\}$.

Recall from Equation 1.7-1 on page 22 in [3] that Bayes' Theorem is given by

$$P[A_j \,|\, B] = \frac{P[A_j \cap B]}{P[B]} = \frac{P[B \,|\, A_j]P[A_j]}{\sum_{i=1}^{n} P[B \,|\, A_i]P[A_i]},$$

where $A_1, A_2, \ldots, A_n$ are mutually disjoint sets such that $\bigcup_{i=1}^{n} A_i = \Omega$. In the context of conditional probabilities, we would like to determine the following quantities (where we have applied Bayes' Theorem since $\{C_i\}$ are mutually disjoint).

$$P[C_1 \,|\, O_2] = \frac{P[O_2 \,|\, C_1]P[C_1]}{\sum_{i=1}^{3} P[O_2 \,|\, C_i]P[C_i]}, \; P[C_3 \,|\, O_2] = \frac{P[O_2 \,|\, C_3]P[C_3]}{\sum_{i=1}^{3} P[O_2 \,|\, C_i]P[C_i]}$$

Note that $P[C_1 \,|\, O_2]$ represents the probability that the car is behind the original door, whereas the $P[C_3 \,|\, O_2]$ corresponds to the probability that the car is behind the remaining door. To solve this problem, we would like to determine which probability is larger (or prove that they are equal).

To evaluate these expressions, we must first determine the simple event and conditional probabilities (which follow directly from the problem statement).

$$P[C_1] = P[C_2] = P[C_3] = \frac{1}{3}$$
$$P[O_2 \,|\, C_1] = \frac{1}{2}, \; P[O_2 \,|\, C_2] = 0, \; P[O_2 \,|\, C_3] = 1$$

Subsituting these expressions, we can obtain estimates of the desired probabilities.

$$P[C_1 \,|\, O_2] = \frac{P[O_2 \,|\, C_1]P[C_1]}{\sum_{i=1}^{3} P[O_2 \,|\, C_i]P[C_i]} = \frac{(1/2)(1/3)}{(1/2)(1/3) + (0)(1/3) + (1)(1/3)} = \frac{1}{3}$$

$$P[C_3 \,|\, O_2] = \frac{P[O_2 \,|\, C_3]P[C_3]}{\sum_{i=1}^{3} P[O_2 \,|\, C_i]P[C_i]} = \frac{(1)(1/3)}{(1/2)(1/3) + (0)(1/3) + (1)(1/3)} = \frac{2}{3}$$

In conclusion, we find that **you should always switch** since $P[C_3 \,|\, O_2] > P[C_1 \,|\, O_2]$. From Bayes' Theorem we find that you will win the car with probability 2/3 if you switch, whereas you will only win with probability 1/3 by staying with the original door. (QED)

Problem 2.1

The event of k successes in n trials (regardless of the order) is the binomial law $b(k; n, p)$. Let $n = 10$ and $p = 0.3$. Define the random variable (r.v.) X by

$$X(k) = \begin{cases} 1, & \text{for } 0 \le k \le 2 \\ 2, & \text{for } 2 < k \le 5 \\ 3, & \text{for } 5 < k \le 8 \\ 4, & \text{for } 8 < k \le 10 \end{cases}.$$

Compute $P[X = j]$ for $j = 1, \ldots, 4$. Plot $F_X(x) = P[X \le x]$ for all x.

First, we recall that the binomial law $b(k; n, p)$ is given by Equation 1.9-1 on page 33 of [3].

$$b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

Consider the event $A_k \triangleq \{k \text{ successes in } n \text{ trials}\}$. Since the $\{A_k\}$ are disjoint (i.e., $\{A_i\} \cap \{A_j\} = \phi$ for $i \neq j$), then we obtain

$$P[X = 1] = P\left[\bigcup_{k=0}^{2} \{A_k\}\right] = \sum_{k=0}^{2} P[\{A_k\}] = \sum_{k=0}^{2} b(k; n, p). \tag{2}$$

Similarly, as shown on page 35 in [3], the probability of more than i successes but no more than j successes in n trials is given by

$$\sum_{k=i+1}^{j} b(k; n, p). \tag{3}$$

Applying Equations 1-3 gives the following values of $P[X = j]$ for $j = 1, \ldots, 4$.

$$\boxed{\begin{aligned} P[X = 1] &= \textstyle\sum_{k=0}^{2} b(k; n, p) \approx 0.382783 \\ P[X = 2] &= \textstyle\sum_{k=3}^{5} b(k; n, p) \approx 0.569868 \\ P[X = 3] &= \textstyle\sum_{k=6}^{8} b(k; n, p) \approx 0.047205 \\ P[X = 4] &= \textstyle\sum_{k=9}^{10} b(k; n, p) \approx 0.000144 \end{aligned}}$$

At this point, we note that $P[X = j] = 0$ for $j \notin \{1, \ldots, 4\}$. As a result, we obtain the following expression

$$F_X(x) = P[X \le x] = \sum_{k=0}^{\lfloor x \rfloor} P[X = k],$$

where $\lfloor x \rfloor$ denotes the largest integer smaller or equal to x. To conclude, $F_X(x)$ is shown in Figure 1.

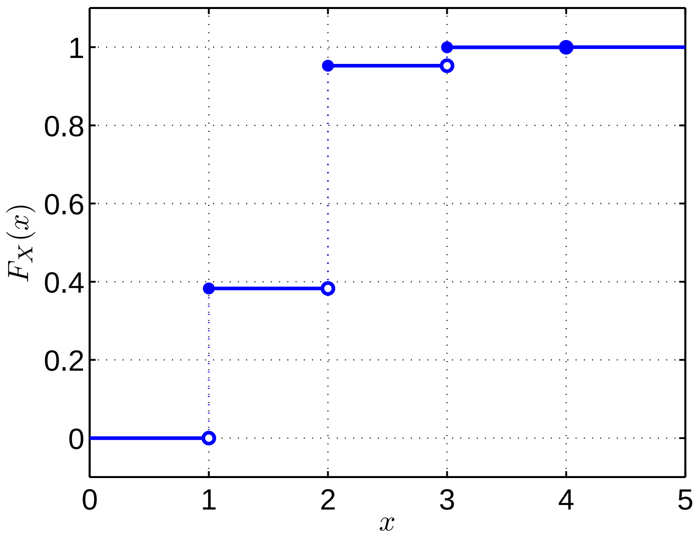


Figure 1: Plot of $F_X(x) = P[X \leq x]$, as derived in Problem 2.1.

Problem 2.2

In a restaurant known for its unusual service, the time X, in minutes, that a customer has to wait before he captures the attention of a waiter is specified by the following distribution function.

$$F_X(x) = \begin{cases} \left(\frac{x}{2}\right)^2, & \text{for } 0 \leq x \leq 1 \\ \frac{x}{4}, & \text{for } 1 \leq x \leq 2 \\ \frac{1}{2}, & \text{for } 2 \leq x \leq 10 \\ \frac{x}{20}, & \text{for } 10 \leq x \leq 20 \\ 1, & \text{for } x \geq 20 \end{cases} \tag{4}$$

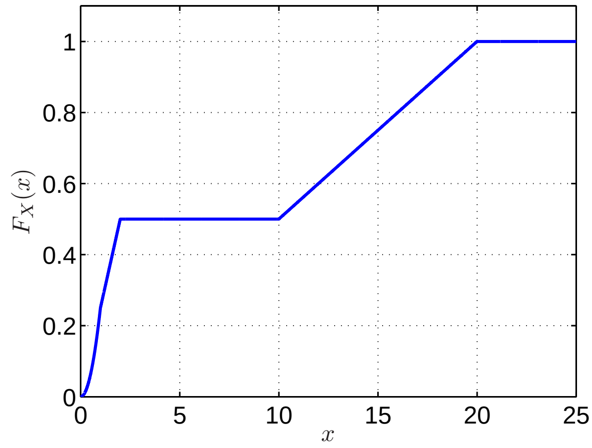
(a) Sketch $F_X(x)$. (b) Compute and sketch the pdf $f_X(x)$. Verify that the area under the pdf is indeed unity. (c) What is the probability that the customer will have to wait (1) at least 10 minutes, (2) less than 5 minutes, (3) between 5 and 10 minutes, and (4) exactly 1 minute?

Part (a)

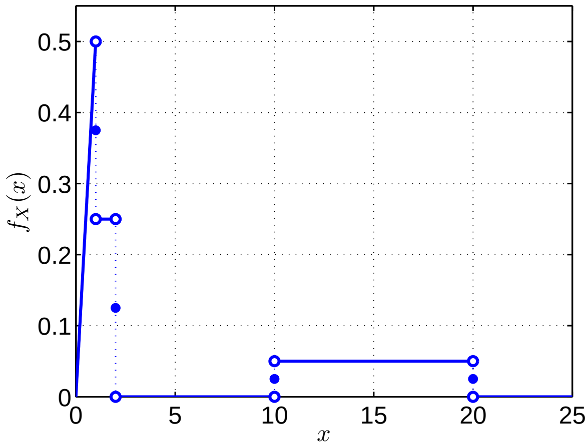
A sketch of $F_X(x)$ is included in Figure 2(a).

Part (b)

Recall that, if $F_X(x)$ is a continuous function of x, then $F_X(x) = F_X(x^-)$ (i.e., the limits taken from the left-hand and right-hand sides are identical). Furthermore, from page 75 in [3], recall that "if $F_X(x)$ is continuous for every x and its derivative exists everywhere except at a countable set of points, then we say that X is a continuous random variable. At points where $F'_X(x)$ exists, the pdf is $f_x(x) = F'_X(x)$. At points where $F_X(x)$ is continuous, but $F'_X(x)$ is discontinuous, we can assign *any* positive number to $f_x(x)$."



(a) $F_x(x)$



(b) $f_x(x)$

Figure 2: Comparison of (a) the probability distribution function (PDF) $F_X(x)$ and (b) the probability density function (pdf) $f_X(x)$, as derived in parts (a) and (b) of Problem 2.2.

From inspection of Equation 4 and Figure 2(a), it is apparent that $F_X(x)$ is a continuous function. As a result, we can apply the previous definition to compute $f(x)$. Note that $F'_X(x)$ is discontinuous for $x \in \{1, 2, 10, 20\}$. Following the definition given in [3], we can specify any value for $f_X(x)$ at these discontinuities. For this problem, we chose to assign the value at a discontinuity ξ using the following expression.

$$f_X(\xi) = \lim_{\varepsilon \to 0} \frac{f_X(\xi + \varepsilon) + f_X(\xi - \varepsilon)}{2}$$

Applying these definitions to $F_X(x)$ gives the following expression for $f_X(x)$.

$$\boxed{f_X(x) = \begin{cases} \frac{x}{2}, & \text{for } 0 \le x < 1 \\ \frac{3}{8}, & \text{for } x = 1 \\ \frac{1}{4}, & \text{for } 1 < x < 2 \\ \frac{1}{8}, & \text{for } x = 2 \\ 0, & \text{for } 2 < x < 10 \\ \frac{1}{40}, & \text{for } x = 10 \\ \frac{1}{20}, & \text{for } 10 < x < 20 \\ \frac{1}{40}, & \text{for } x = 20 \\ 0, & \text{for } x > 20 \end{cases}} \tag{5}$$

Since waiting time must be a nonnegative number, the area under the pdf is given by the following expression.

$$\int_0^\infty f_X(\xi)\, d\xi = \frac{1}{2}\int_0^1 \xi\, d\xi + \frac{1}{4}\int_1^2 d\xi + \frac{1}{20}\int_{10}^{20} d\xi = \frac{1}{4} + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{20}{20} - \frac{10}{20}\right) = 1$$

In conclusion, we verify that the area under $f_X(x)$ is indeed unity. (QED)

Part (c)

From Equation 2.3-1 on page 62 in [3], we recall that $F_X(x)$ is defined by

$$F_X(x) = P[\{\zeta : X(\zeta) \leq x\}] = P_X[(-\infty, x]].$$

Applying this definition, we find that the probability that the customer will have to wait at least 10 minutes is given by the following expression.

$$P[X \geq 10] = P[X \leq \infty] - P[X \leq 10] + P[X = 10] = F_X(\infty) - F_X(10) = 1 - \frac{1}{2} = \frac{1}{2}$$

Note that in the previous expression, we used the fact that $F_X(\infty) = 1$ (i.e., that probability of the certain event is unity). In addition, we recall that $P[X = x]$ is zero whenever $F_X(x)$ is continuous. Similarly, the probability that the customer will have to wait less than 5 minutes is

$$P[X < 5] = P[X \leq 5] - P[X = 5] = F_X(5) = \frac{1}{2}.$$

The probability that the customer will have to wait between 5 and 10 minutes is

$$P[5 \leq X \leq 10] = P[X \leq 10] - P[X \leq 5] + P[X = 5] = F_X(10) - F_X(5) = \frac{1}{2} - \frac{1}{2} = 0.$$

Finally, the probability that the customer will have to wait exactly 1 minute is zero, since $P[X = x]$ is zero whenever $F_X(x)$ is continuous. In conclusion, the resulting probabilities for each waiting period are tabulated below.

probability of waiting at least 10 minutes :	$P[X \geq 10] = \frac{1}{2}$
probability of waiting less than 5 minutes :	$P[X < 5] = \frac{1}{2}$
probability of waiting between 5 and 10 minutes :	$P[5 \leq X \leq 10] = 0$
probability of waiting exactly 1 minute :	$P[X = 1] = 0$

Problem 2.7

Write the probability density functions (using delta functions) for (a) the Bernoulli, (b) the binomial, and (c) the Poisson probability mass functions (PMF's).

Part (a)

Recall from Equation 2.5-7 on page 76 in [3] that the PDF of a discrete r.v. is defined as

$$F_X(x) \triangleq P[X \leq x] = \sum_{\text{all } x_i \leq x} P_X(x_i).$$

Expressing this equation in terms of the unit step $u(x - x_i)$, we obtain Equation 2.5-12a.

$$F_X(x) = \sum_{i=\infty}^{\infty} P_X(x_i) u(x - x_i)$$

Finally, using the Dirac delta function $\delta(x - x_i)$, we find the general form for the discrete r.v. pdf.

$$f_X(x) = \frac{dF_X(x)}{dx} = \sum_{i=\infty}^{\infty} P_X(x_i)\delta(x - x_i) \tag{6}$$

At this point, we also recall that the PMF of a Bernoulli random variable is given by

$$P_X(x) = \begin{cases} p, & \text{for } x = 0 \\ 1 - p, & \text{for } x = 1 \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $0 < p < 1$ [3]. Substituting Equation 7 into Equation 6 gives the desired expression for the pdf of a Bernoulli random variable.

$$\boxed{f_X(x) = p\,\delta(x) + (1 - p)\,\delta(x - 1)}$$

Part (b)

From Equation 2.5-10 in [3], recall that the PMF of a binomial random variable is given by

$$P_X(k) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1 - p)^{n-k}, & \text{for } k = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where n is a positive integer and $0 < p < 1$. Substituting Equation 8 into Equation 6 gives the desired expression for the pdf of a binomial random variable.

$$\boxed{f_X(x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1 - p)^{n-k} \delta(x - k)}$$

Part (c)

From Equation 2.5-11 in [3], recall that the PMF of a Poisson random variable, with $a > 0$, is given by the following expression.

$$P_X(k) = \begin{cases} e^{-a} \frac{a^k}{k!}, & \text{for } k = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Substituting Equation 9 into Equation 6 yields the pdf of a Poisson random variable.

$$\boxed{f_X(x) = e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!}\, \delta(x - k)}$$

Note that e^{-a} is a constant and can be brought outside of the summation. In addition, we observe that the expressions derived in parts (b) and (c) agree with the results presented on page 79 of [3].

Problem 2.11

The time-to-failure in months, X, of light bulbs produced at two manufacturing plants A and B obey, respectively, the following PDF's.

$$F_X(x) = (1 - e^{-x/5})u(x) \text{ for plant } A$$
$$F_X(x) = (1 - e^{-x/2})u(x) \text{ for plant } B$$

Plant B produces three times as many bulbs as plant A. The bulbs, indistinguishable to the eye, are intermingled and sold. What is the probability that a bulb purchased at random will burn at least (a) two months, (b) five months, and (c) seven months?

This problem can be solved by evaluating the unconditional PDF that represents the "average" time-to-failure. Recall that the probability of an event E, in terms of n mutually exclusive and exhaustive events $\{A_i\}$ for $i = 1, \ldots, n$, is given by the following expression [3].

$$F_X(x) = \sum_{i=1}^{n} F_X(x|A_i)P[A_i]$$

Since we can enumerate the sample space as $\Omega = \{A, B\}$, where A and B represent the manufacturing plants, we can obtain the "average" PDF directly from the provided conditional PDF's.

$$F_X(x) = F_X(x|A)P[A] + F_X(x|B)P[B] = \frac{1}{4}(1 - e^{-x/5})u(x) + \frac{3}{4}(1 - e^{-x/2})u(x)$$

Note that in the previous expression we have applied the result that $P[A] = 1/4$ and $P[B] = 3/4$, since plant B manufactures three times as many bulbs and plant A. To determine the probability that a bulb purchased at random will burn at least two months, five months, and seven months, we can substitute $x = 2$, $x = 5$, and $x = 7$, respectively.

$$\begin{aligned}
\text{probability of at least two months}: &\quad 1 - F_X(2) = 1 - \tfrac{1}{4}(1 - e^{-2/5}) - \tfrac{3}{4}(1 - e^{-2/2}) \approx 0.44349 \\
\text{probability of at least five months}: &\quad 1 - F_X(5) = 1 - \tfrac{1}{4}(1 - e^{-5/5}) - \tfrac{3}{4}(1 - e^{-5/2}) \approx 0.15353 \\
\text{probability of at least seven months}: &\quad 1 - F_X(7) = 1 - \tfrac{1}{4}(1 - e^{-7/5}) - \tfrac{3}{4}(1 - e^{-7/2}) \approx 0.08430
\end{aligned}$$

Note that, in the previous expression, we have applied the fact that $F_X(\infty) = 1$ to obtain

$$P[X \geq x] = F_X(\infty) - F_X(x) + P[X = x] = 1 - F_X(x).$$

Problem 2.12

Show that the conditional distribution of X given the event $A = \{b < X \leq a\}$ is

$$F_X(x|A) = \begin{cases} 0, & \text{for } x \leq b \\ \frac{F_X(x)-F_X(b)}{F_X(a)-F_X(b)}, & \text{for } b < x \leq a \\ 1, & \text{for } x \geq a. \end{cases}$$

Recall from Equation 2.6-1 in [3] that the conditional distribution function of X given event A is

$$F_X(x|A) = \frac{P[X \leq x, A]}{P[A]}, \tag{10}$$

where $P[X \leq x, A]$ is the probability of the joint event $\{X \leq x\} \cap A$ and $P[A] \neq 0$. First, let's consider the case where $x \leq b$; in this case, the joint event is given by the following expression.

$$\{X \leq x\} \cap A = \{X \leq x\} \cap \{b < X \leq a\} = \phi, \text{ for } x \leq b$$

In other words, we find that the joint event is the empty set ϕ. Since $\Omega \cap \phi = \phi$, we find that $P[X \leq x, A] = P[\phi] = 0$, for $x \leq b$. Substituting into Equation 10 gives the following condition.

$$F_X(x|A) = \frac{P[X \leq x, A]}{P[A]} = \frac{P[\phi]}{P[A]} = 0, \text{ for } x \leq b \tag{11}$$

Now let's consider the case where $x \geq a$; in this case, the joint event is given by

$$\{X \leq x\} \cap A = \{X \leq x\} \cap \{b < X \leq a\} = \{b < X \leq a\} = A, \text{ for } x \geq a.$$

In other words, we find that the joint event is the event A itself (since $A \subset \{X \leq x\}$, for $x \geq a$). As a result, we can conclude that $P[X \leq x, A] = P[A]$ for $x \geq a$, which gives the following condition.

$$F_X(x|A) = \frac{P[X \leq x, A]}{P[A]} = \frac{P[A]}{P[A]} = 1, \text{ for } x \geq a \tag{12}$$

Finally, let's consider the remaining case in which $b < x \leq a$. Similar to the previous cases, the joint event can be determined as follows.

$$\{X \leq x\} \cap A = \{X \leq x\} \cap \{b < X \leq a\} = \{b < X \leq x\}, \text{ for } b < x \leq a$$

In this case, the probability of the joint event is $P[X \leq x, A] = P[b < X \leq x]$. Before we proceed, recall the following expression given by Equation 2.3-2 in [3].

$$P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1) \tag{13}$$

Substituting for the joint event in Equation 10 and applying Equation 13 gives the final condition.

$$F_X(x|A) = \frac{P[X \leq x, A]}{P[A]} = \frac{P[b < X \leq x]}{P[b < X \leq a]} = \frac{F_X(x) - F_X(b)}{F_X(a) - F_X(b)}, \text{ for } b < x \leq a \tag{14}$$

Combining the conditions given by Equations 11, 12, and 14 gives the desired result.

$$\boxed{F_X(x|A) = \begin{cases} 0, & \text{for } x \leq b \\ \frac{F_X(x)-F_X(b)}{F_X(a)-F_X(b)}, & \text{for } b < x \leq a \\ 1, & \text{for } x \geq a. \end{cases}}$$

(QED)

Problem 2.13

The number of people Y waiting in a bank queue obeys the Poisson law as

$$P[Y=k|X=x]=e^{-x}\frac{x^k}{k!},\ k\geq 0, x>0,$$

since the normalized serving time of the teller x is constant. However, the serving time can be more accurately modeled as a r.v. X. For simplicity, let X be a uniform r.v. with

$$f_X(x)=\frac{1}{5}[u(x)-u(x-5)]. \tag{15}$$

Then $P[Y=k|X=x]$ is still Poisson, but $P[Y=k]$ isn't. Compute $P[Y=k]$ for integer $k\geq 0$.

First, we recall the following result from page 84 in [3]. For the events $B=\{Y=k\}$ and $\{X=x\}$ defined on the same probability space, the average unconditional probability is given by

$$P[Y=k]=\int_{-\infty}^{\infty}P[Y=k|X=x]f_X(x)dx,$$

where $f_X(x)$ is the pdf for the random variable X. If we assume, for any given outcome $\{X=x\}$, the probability $P[Y=k|X=x]$ is Poisson, then we obtain the following unconditional PMF.

$$P[Y=k]=\int_0^{\infty}e^{-x}\frac{x^k}{k!}f_X(x)dx,\ \text{for } k\geq 0 \tag{16}$$

Note that Equation 16 is also known as the *Poisson transform.* Substituting Equation 15 into Equation 16 gives following expression for $P[Y=k]$.

$$P[Y=k]=\frac{1}{5}\int_0^{\infty}e^{-x}\frac{x^k}{k!}[u(x)-u(x-5)]dx$$

Note that $[u(x)-u(x-5)]$ is non-zero and equal to unity only on the interval $0\leq x<5$. As a result, we can simplify the previous expression as

$$P[Y=k]=\frac{1}{5k!}\int_0^{5}e^{-x}x^k dx=\frac{\gamma(k+1,5)}{5k!}, \tag{17}$$

where $\gamma(\alpha,\beta)$ is the lower incomplete gamma function given by [4]; recall from the same reference that $\gamma(\alpha,\beta)$ has the following simple form for integer α.

$$\gamma(\alpha,\beta)\triangleq\int_0^{\beta}e^{-x}x^{\alpha-1}dx=(\alpha-1)!\left(1-e^{-\beta}\sum_{n=0}^{\alpha-1}\frac{\beta^n}{n!}\right),\ \text{for } \alpha\in\mathbb{Z} \tag{18}$$

Substituting Equation 18 into Equation 17 gives the desired expression for $P[Y=k]$.

$$\boxed{P[Y=k]=\frac{1}{5}\left(1-e^{-5}\sum_{n=0}^{k}\frac{5^n}{n!}\right),\ \text{for integer } k\geq 0}$$

Note that an alternate method to derive this solution is to integrate Equation 17 by parts – leading to the expansion for integer k presented in Equation 18.

Problem 2.29

Use the basic properties of the joint PDF $F_{XY}(x,y)$ to show

(a) $f_{XY}(x,y)dx\,dy = P[x < X \leq x+dx, y < Y \leq y+dy]$,

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dx\,dy = 1$,

(c) $f_{XY}(x,y) \geq 0$.

Part (a)

Recall that, from the basic properties of the joint PDF $F_{XY}(x,y)$, we can derive the following relationship presented on page 92 of [3].

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$

Substituting $\{x_1 = x, x_2 = x+\Delta x\}$ and $\{y_1 = y, y_2 = y+\Delta y\}$ gives the following expression.

$$\begin{aligned} &P[x < X \leq x+\Delta x, y < Y \leq y+\Delta y] \\ &\quad = F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x, y+\Delta y) - F_{XY}(x+\Delta x, y) + F_{XY}(x,y) \end{aligned} \tag{19}$$

At this point, we recall that the partial derivative has the following general form [5].

$$\frac{\partial f}{\partial x_m} = \lim_{\Delta x_m \to 0} \frac{f(x_1,\ldots,x_m+\Delta x_m,\ldots,x_n) - f(x_1,\ldots,x_m,\ldots,x_n)}{\Delta x_m}$$

For the two-dimensional function $f(x,y)$ defined on $\mathbb{R}^2$, we have the following result.

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Evaluating the partial derivative of this expression, with respect to y, gives the following general form for the mixed partial derivative.

$$\frac{\partial f(x,y)}{\partial x\,\partial y} = \lim_{\Delta y \to 0}\left\{ \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x\,\Delta y} - \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x\,\Delta y} \right\}$$

$$\Rightarrow \frac{\partial f(x,y)}{\partial x\,\partial y} = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x,y)}{\Delta x\,\Delta y} \tag{20}$$

If we divide both sides of Equation 19 by $\Delta x\,\Delta y$ and evaluate the limit as $\Delta x \to 0$ and $\Delta y \to 0$, then we can derive the following relationship by applying Equation 20.

$$\begin{aligned} &\lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{P[x < X \leq x+\Delta x, y < Y \leq y+\Delta y]}{\Delta x\,\Delta y} \\ &\quad = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x, y+\Delta y) - F_{XY}(x+\Delta x, y) + F_{XY}(x,y)}{\Delta x\,\Delta y} \\ &\quad = \frac{\partial F_{XY}(x,y)}{\partial x\,\partial y} \triangleq f_{XY}(x,y) \end{aligned} \tag{21}$$

Multiplying both sides of this expression by $\Delta x\,\Delta y$ and considering the limiting behavior for small Δx and Δy, gives the desired property of the joint pdf $f_{XY}(x,y)$. (QED)

$$\therefore \boxed{f_{XY}(x,y)dx\,dy = P[x < X \leq x+dx, y < Y \leq y+dy]}$$

Part (b)

As in the previous part, we can derive the desired property of the joint pdf $f_{XY}(x, y)$ by applying known properties of the joint PDF $F_{XY}(x, y)$. Recall the following definition of $f_{XY}(x, y)$ from Equation 2.6-22 on page 89 in [3].

$$f_{XY}(x, y) \triangleq \frac{\partial F_{XY}(x, y)}{\partial x \, \partial y}$$

Integrating this expression with respect to x and y yields the following result.

$$F_{XY}(x, y) = \int_{-\infty}^{x} d\xi \int_{-\infty}^{y} d\eta f_{XY}(\xi, \eta)$$

Substituting $\{x = \infty, \, y = \infty\}$ yields the following relationship.

$$F_{XY}(\infty, \infty) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta f_{XY}(\xi, \eta)$$

At this point we recall the basic property that $F_{XY}(\infty, \infty) = 1$, as defined on page 90 in [3]. Substituting this result into the previous expression gives the desired property of the joint pdf $f_{XY}(x, y)$. (QED)

$$\therefore \boxed{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx \, dy = 1}$$

Part (c)

Recall that, for the probability space $\mathcal{P}$ (with sample space Ω, σ-field $\mathcal{F}$, and probability measure P), the event $E \in \mathcal{F}$ has a probability $P[E] \geq 0$ (i.e., Axiom 1 on page 11 in [3]). As a result, for positive Δx and Δy, the following expression must hold.

$$P[x < X \leq x + \Delta x, y < Y \leq y + \Delta y] \geq 0$$

Dividing both sides of this expression by the positive quantity $\Delta x \, \Delta y$ and taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives the following inequality.

$$\lim_{\Delta x \to 0, \, \Delta y \to 0} \frac{P[x < X \leq x + \Delta x, y < Y \leq y + \Delta y]}{\Delta x \, \Delta y} \geq 0$$

At this point we recognize the the left-hand side of this inequality is identical to Equation 21. As a result, we can conclude that the desired property of the joint pdf $f_{XY}(x, y)$ must be valid. (QED)

$$f_{XY}(x, y) = \lim_{\Delta x \to 0, \, \Delta y \to 0} \frac{P[x < X \leq x + \Delta x, y < Y \leq y + \Delta y]}{\Delta x \, \Delta y} \geq 0$$

$$\therefore \boxed{f_{XY}(x, y) \geq 0}$$

Problem 2.30

(a) Show that Equation 2.6-46 factors as $f_X(x)f_Y(y)$ when $\rho = 0$. What are $f_X(x)$ and $f_Y(y)$? (b) For $\sigma = 1$ and $\rho = 0$, what is $P[-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}]$?

Part (a)

First, we recall that Equation 2.6-46 describes the *bivariate normal distribution* (also known as the jointly Gaussian probability density of two random variables).

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2\sigma^2(1-\rho^2)}(x^2+y^2-2\rho xy)\right) \tag{22}$$

Substituting $\rho = 0$ in Equation 22 gives the following simplified expression.

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2\pi\sigma^2} \exp\left(\frac{-1}{2\sigma^2}(x^2+y^2)\right) \\ &= \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right] \times \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)\right] \end{aligned}$$

We find that for $\rho = 0$ Equation 22 factors as a product of the marginal densities $f_X(x)$ and $f_Y(y)$. In this situation, **the marginal densities have the form of a univariate Normal distribution**. (QED)

$$\therefore \boxed{\begin{aligned} f_{XY} &= f_X(x)f_Y(y), \text{where} \\ f_X(x) &= \tfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{x^2}{2\sigma^2}\right) \\ f_Y(y) &= \tfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{y^2}{2\sigma^2}\right) \end{aligned}}$$

Part (b)

Recall, from Equation 2.6-30 on page 94 in [3], that the probability of any event of the form $\{(X,Y) \in \mathcal{A}\}$ can be computed by the following formula.

$$P[(X,Y) \in \mathcal{A}] = \iint_{\mathcal{A}} f_{XY}(x,y)dx\,dy \tag{23}$$

Also recall that the *error function* is typically encountered when integrating normal distributions.

$$\text{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{\xi^2}{2}} d\xi \tag{24}$$

Applying Equations 23 and 24 to the region $\mathcal{A} = \{-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}\}$ gives the following.

$$\begin{aligned} P\left[-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}\right] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_X(x; \sigma=1, \rho=0) f_Y(y; \sigma=1, \rho=0) dx\,dy \\ &= \left[\frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{x^2}{2}} dx\right] \times \left[\frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{y^2}{2}} dy\right] \\ &= \left[2\,\text{erf}\left(\frac{1}{2}\right)\right] \times \left[2\,\text{erf}\left(\frac{1}{2}\right)\right] \end{aligned}$$

In conclusion, we report the following probability for the event $\{-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}\}$.

$$\boxed{P\left[-\frac{1}{2} < X \leq \frac{1}{2}, -\frac{1}{2} < Y \leq \frac{1}{2}\right] = 4\,\mathrm{erf}^2\left(\frac{1}{2}\right) \approx 0.14663}$$

References

[1] Geoffrey Grimmett and David Stirzaker. *Probability and Random Processes (Third Edition).* Oxford University Press, 2001.

[2] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Cambridge University Press, 2005.

[3] Henry Stark and John W. Woods. *Probability and Random Processes with Applications to Signal Processing (Third Edition).* Prentice-Hall, 2002.

[4] Eric W. Weisstein. Incomplete gamma function. `http://mathworld.wolfram.com/IncompleteGammaFunction.html`.

[5] Eric W. Weisstein. Partial derivative. `http://mathworld.wolfram.com/PartialDerivative.html`.