EN 202: Problem Set 5

Douglas R. Lanman
8 March 2006

Problem 1

Classify each of the following equations as elliptic, parabolic, or hyperbolic. Find and sketch the characteristics (where they exist).

a. $2u_{xx} + 2u_{xy} + 3u_{yy} = 0$
b. $u_{xx} + 2u_{xy} + u_{yy} = 0$
c. $e^{2x}u_{xx} - u_{yy} = 0$
d. $xu_{xx} + u_{yy} = 0$

Recall from class on 2/24/06 that a general linear second-order PDE can be expressed as

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

where $\{a, b, c, d, e, f, g\}$ can all depend on x and y. As was shown in class, the characteristics, in the form $y(x)$, are given by the solution(s) to the following ODE

$$a\frac{dy}{dx}^2 - b\frac{dy}{dx} + c = 0 \tag{1}$$

Further recall that a second-order PDE is classified according to the value of its discriminant as follows.

$$\begin{aligned} b^2 - 4ac > 0: &\quad \text{hyperbolic} \\ b^2 - 4ac = 0: &\quad \text{parabolic} \\ b^2 - 4ac < 0: &\quad \text{elliptic} \end{aligned}$$

Part (a)

For this problem we have $\{a = 2, b = 2, c = 3\}$. We begin by classifying the PDE as follows.

$$\boxed{b^2 - 4ac = -20 < 0 \Rightarrow \text{elliptic}} \tag{2}$$

Since this PDE is elliptic, there can be no real characteristics. We can prove this by substituting into Equation 1.

$$2\frac{dy}{dx}^2 - 2\frac{dy}{dx} + 3 = 0$$

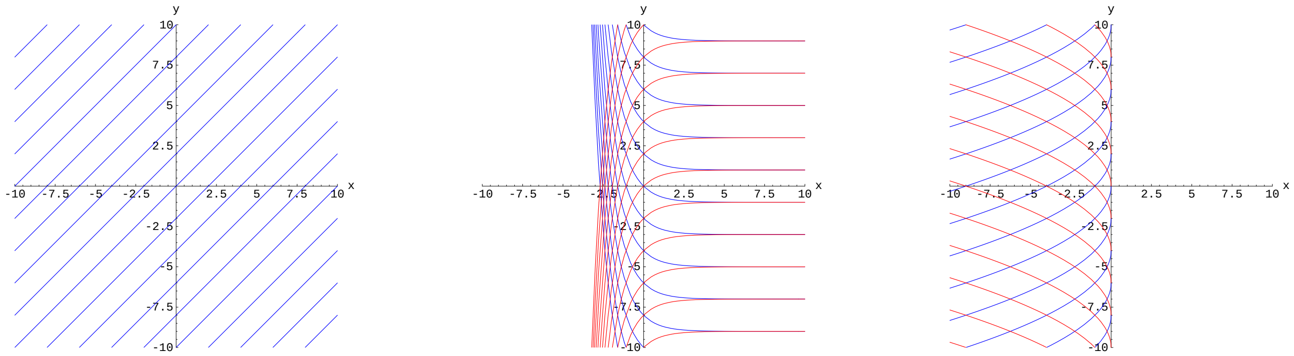
Applying the quadratic formula to solve for dy/dx, we find

$$\frac{dy}{dx} = \frac{2 \pm \sqrt{-20}}{4} = \frac{1}{2}\left(1 \pm i\sqrt{5}\right)$$

Integrating this expression with respect to x, we obtain the solution for the characteristics

$$\boxed{\text{characteristics: } y = \frac{1}{2}\left(1 + i\sqrt{5}\right)x + \xi \text{ and } y = \frac{1}{2}\left(1 - i\sqrt{5}\right)x + \eta} \tag{3}$$

for arbitrary complex-valued constants $\{\xi, \eta\}$. Note that, since these characteristics are complex-valued, we cannot sketch them in $\mathbb{R}^2$.



(a) characteristics for part (b) (b) characteristics for part (c) (c) characteristics for part (d)

Figure 1: Sketch of characteristics found in Problem 1. For subfigures (b) and (c), there are two sets of characteristics shown in red and blue, respectively.

Part (b)

For this problem we have $\{a = 1, b = 2, c = 1\}$. We begin by classifying the PDE as follows.

$$\boxed{b^2 - 4ac = 0 \Rightarrow \text{parabolic}} \tag{4}$$

Since this PDE is parabolic, we expect a single set of characteristics. We can prove this by substituting into Equation 1.

$$\frac{dy}{dx}^2 - 2\frac{dy}{dx} + 1 = 0$$

Applying the quadratic formula to solve for dy/dx, we find

$$\frac{dy}{dx} = 1$$

Integrating this expression with respect to x, we obtain the solution for the characteristics

$$\boxed{\text{characteristics: } y = x + \eta, \text{ for } \eta \in \mathbb{R}} \tag{5}$$

In conclusion, we find a single set of characteristics indexed by the parameter η. This family of lines is sketched in Figure 1(a).

Part (c)

For this problem we have $\{a = e^{2x}, b = 0, c = -1\}$. We begin by classifying the PDE as follows.

$$\boxed{b^2 - 4ac = 4e^{2x} > 0, \forall x \Rightarrow \text{hyperbolic}} \tag{6}$$

Since this PDE is hyperbolic, we expect two sets of characteristics. We can prove this by substituting into Equation 1.

$$e^{2x}\frac{dy}{dx}^2 - 1 = 0$$

Applying the quadratic formula to solve for dy/dx, we find

$$\frac{dy}{dx} = \pm e^{-x}$$

Integrating this expression with respect to x, we obtain the solution for the characteristics

$$\boxed{\text{characteristics: } y = e^{-x} + \xi \text{ and } y = -e^{-x} + \eta} \tag{7}$$

for arbitrary real-valued constants $\{\xi, \eta\}$. In conclusion, we find two sets of characteristics. These two families of lines are sketched in Figure 1(b). Note that, in the figure, the blue lines represent the characteristics indexed by ξ and the red lines represent those indexed by η.

Part (d)

For this problem we have $\{a = x, b = 0, c = 1\}$. We begin by classifying the PDE as follows.

$$\boxed{b^2 - 4ac = -4x \Rightarrow \begin{cases} \text{elliptic for } x > 0 \\ \text{parabolic for } x = 0 \\ \text{hyperbolic for } x < 0 \end{cases}} \tag{8}$$

Notice that the discriminant is a function of x. As a result, the classification of the PDE varies by region. Regardless, we can solve for the characteristics as before. Substituting into Equation 1 we find

$$x \frac{dy}{dx}^2 + 1 = 0$$

Applying the quadratic formula to solve for dy/dx, we find

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x}}{2x} = \pm i x^{-\frac{1}{2}}$$

Integrating this expression with respect to x, we obtain the solution for the characteristics

$$\boxed{\text{characteristics: } y = \pm(2i)\sqrt{x} + C} \tag{9}$$

for an arbitrary, possibly complex-valued, constant C. Notice that, in the elliptic region ($x > 0$), the characteristics will be complex-valued. For $x = 0$, the solutions are given by the set of all lines parallel to the x-axis. Finally, in the hyperbolic region ($x < 0$), there are two sets of real-valued characteristics given by

$$y = 2\sqrt{|x|} + \xi \text{ and } y = -2\sqrt{|x|} + \eta$$

for arbitrary real-valued constants $\{\xi, \eta\}$. These two families of lines are sketched in Figure 1(c). Note that, in the figure, the blue lines represent the characteristics indexed by η and the red lines represent those indexed by ξ.

Problem 2

Consider the initial value problem:

$$12u_{xx} - u_{xy} - u_{yy} = 0,\ u(x,0) = u_0(x),\ u_y(x,0) = \nu_0(x),\ \text{for } -\infty < x < \infty$$

a. Show that $\xi = x + 3y$, $\eta = x - 4y$ are characteristic coordinates.

b. Using the characteristic coordinates, find the general solution to the IVP.

c. Illustrate your answer for $u(x,0) = u_0(x) = e^{-\frac{1}{2}x^2}$, $u_y(x,0) = \nu_0(x) = 0$

Part (a)

We can proceed as in Problem 1 by solving for the characteristics using Equation 1. For this problem, we have $\{a = 12, b = -1, c = -1\}$. Note that the discriminant $b^2 - 4ac = 49 > 0$, so this is a hyperbolic PDE. As a result, there should be two sets of characteristics. Substituting into Equation 1 we find

$$12\frac{dy}{dx}^2 + \frac{dy}{dx} - 1 = 0$$

Applying the quadratic formula to solve for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{-1 \pm 7}{24}$$

Integrating this expression with respect to x gives the solution for the characteristics

$$y = \left(\frac{-1 \pm 7}{24}\right) x + C$$

for an arbitrary real-valued constant C. If we consider the positive and negative terms separately, we find

$$y = \frac{1}{4}x + C_1 \text{ and } y = -\frac{1}{3}x + C_2$$

Rearranging terms gives

$$\Rightarrow -\frac{1}{4}x + y = C_1 \text{ and } \frac{1}{3}x + y = C_2$$

Multiplying the first equation by -4 and the second by 3 gives

$$\Rightarrow x - 4y = C_1' \text{ and } x + 3y = C_2'$$

Note that, since $\{C_1', C_2'\}$ are arbitrary real-valued constants, we can express the characteristics in the desired form

$$\boxed{\text{characteristics: } \xi = x + 3y \text{ and } \eta = x - 4y} \tag{10}$$

for $\xi = C_2'$ and $\eta = C_1'$. (QED)

Part (b)

As discussed in class on 2/24/06, we can use the characteristic coordinates to find the general solution to the IVP. First, note that the general solution $u(x, y)$ can be written using characteristic coordinates in the familiar form $U(\xi, \eta) = u(\bar{x}(\xi, \eta), \bar{y}(\xi, \eta))$. We can apply the chain rule to obtain a PDE in (ξ, η)-coordinates. First, consider u_x.

$$u_x = \frac{d}{dx}U(\xi, \eta) = U_\xi \frac{d\xi}{dx} + U_\eta \frac{d\eta}{dx} = U_\xi + U_\eta$$

Notice that we have applied the result that $d\xi/dx = 1$ and $d\eta/dx = 1$, obtained by differentiating the expressions found in Part (a). Next, consider u_{xx}.

$$u_{xx} = \frac{d^2}{dx^2}U(\xi, \eta) = U_{\xi\xi}\frac{d\xi}{dx} + U_{\xi\eta}\frac{d\eta}{dx} + U_{\eta\xi}\frac{d\xi}{dx} + U_{\eta\eta}\frac{d\eta}{dx}$$
$$\Rightarrow u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

The mixed derivative term u_{xy} is given by the following expression.

$$u_{xy} = \frac{d^2}{dxdy}U(\xi, \eta) = U_{\xi\xi}\frac{d\xi}{dy} + U_{\xi\eta}\frac{d\eta}{dy} + U_{\eta\xi}\frac{d\xi}{dy} + U_{\eta\eta}\frac{d\eta}{dy}$$
$$\Rightarrow u_{xy} = 3U_{\xi\xi} - U_{\xi\eta} - 4U_{\eta\eta}$$

Notice that we have applied the result that $d\xi/dy = 3$ and $d\eta/dy = -4$, obtained by differentiating the expressions found in Part (a). Similarly, u_y is given by the following expression.

$$u_y = \frac{d}{dy}U(\xi, \eta) = U_\xi \frac{d\xi}{dy} + U_\eta \frac{d\eta}{dy} = 3U_\xi - 4U_\eta$$

Finally, we can evaluate u_{yy}.

$$u_{yy} = \frac{d^2}{dy^2}U(\xi, \eta) = 3\left(U_{\xi\xi}\frac{d\xi}{dy} + U_{\xi\eta}\frac{d\eta}{dy}\right) - 4\left(U_{\eta\xi}\frac{d\xi}{dy} + U_{\eta\eta}\frac{d\eta}{dy}\right)$$

Substituting $d\xi/dy = 3$ and $d\eta/dy = -4$, we find

$$u_{yy} = \frac{d^2}{dy^2}U(\xi, \eta) = 9U_{\xi\xi} - 24U_{\xi\eta} + 16U_{\eta\eta}$$

At this point we can substitute back into the original PDE to obtain a PDE in (ξ, η)-coordinates.

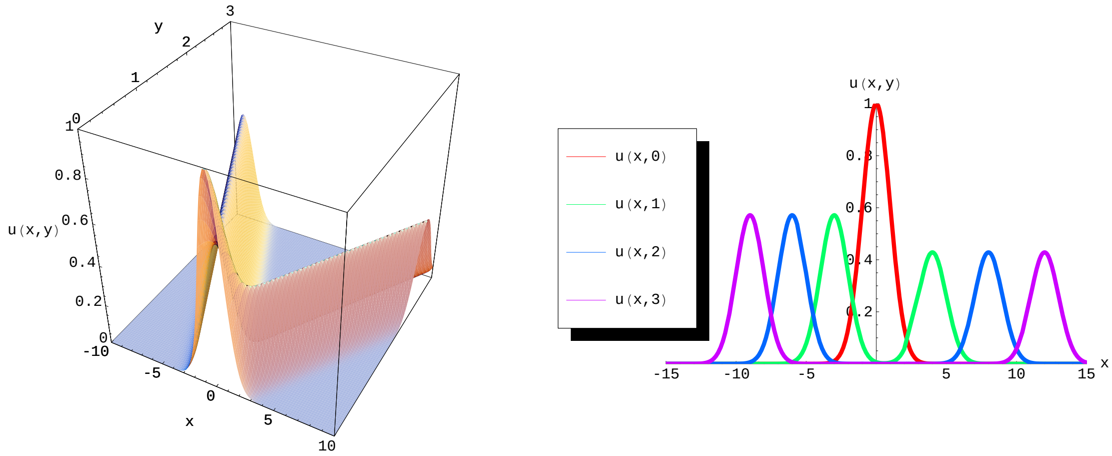
$$12u_{xx} - u_{xy} - u_{yy} = 0$$
$$\Rightarrow 12\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) - \left(3U_{\xi\xi} - U_{\xi\eta} - 4U_{\eta\eta}\right) - \left(9U_{\xi\xi} - 24U_{\xi\eta} + 16U_{\eta\eta}\right) = 0$$

Simplifying, we find

$$U_{\xi\eta} = 0 \Rightarrow U(\xi, \eta) = g(\xi) + h(\eta) \Rightarrow u(x, y) = g(x + 3y) + h(x - 4y)$$

Note that, since the mixed derivative is equal to zero, the solution can be composed of two independent terms (see class notes on 2/27/06). We can proceed by substituting for the initial conditions on Γ.

$$u(x, 0) = g(x) + h(x) = u_0(x) \tag{11}$$



(a) solution surface for $y \geq 0$

(b) plot of $u(x, y)$ for several times y

Figure 2: Illustration of solution $u(x, y)$ found in Problem 2.

$$u_y(x, y) = 3g'(x + 3y) - 4h'(x - 4y)$$
$$\Rightarrow u_y(x, 0) = 3g'(x) - 4h'(x) = \nu_0(x)$$

Integrating this expression with respect to x gives

$$3g(x) - 4h(x) = \int_{-\infty}^{x} \nu_0(s)ds + C, \text{ for } C \in \mathbb{R} \tag{12}$$

Combining Equations 11 and 12, we can obtain expressions for $g(x, y)$ and $h(x, y)$.

$$g(x) = \frac{4}{7}u_0(x) + \frac{1}{7}\int_{-\infty}^{x} \nu_0(s)ds + C$$

$$h(x) = \frac{3}{7}u_0(x) - \frac{1}{7}\int_{-\infty}^{x} \nu_0(s)ds - C$$

Finally, we can substitute back into the expression for $u(x, y)$ obtained previously.

$$u(x, y) = \left[\frac{4}{7}u_0(x + 3y) + \frac{1}{7}\int_{-\infty}^{x+3y} \nu_0(s)ds + C\right] + \left[\frac{3}{7}u_0(x - 4y) - \frac{1}{7}\int_{-\infty}^{x-4y} \nu_0(s)ds - C\right]$$

Notice that the constant term C cancels out. In addition, the integrals can be combined, yielding the general solution to the IVP.

$$\boxed{u(x, y) = \left[\frac{4}{7}u_0(x + 3y) + \frac{3}{7}u_0(x - 4y)\right] + \frac{1}{7}\int_{x-4y}^{x+3y} \nu_0(s)ds} \tag{13}$$

Part (c)

Notice, for the initial conditions $u_0(x) = e^{-\frac{1}{2}x^2}$ and $\nu_0(x) = 0$, the integrand is zero. Substituting into Equation 13, we find the solution to this specific IVP.

$$\boxed{u(x, y) = \frac{4}{7}e^{-\frac{1}{2}(x+3y)^2} + \frac{3}{7}e^{-\frac{1}{2}(x-4y)^2}} \tag{14}$$

The solution surface $u(x, y)$ is illustrated in Figure 2(a) for $y \geq 0$. In addition, the solution is plotted for several instants in time y in Figure 2(b).

Problem 3

An initial value problem for Problem 1(c) above is $e^{2x}u_{xx} - u_{yy} = 0$ with $u(x, 0) = u_0(x)$ and $u_y(x, 0) = 0$, for $-\infty < x < \infty$. Use the characteristics to transform the PDE into an equation for $U(\xi, \eta)$. **You need not solve the equation or find a solution to this IVP**.

Recall from Problem 1(c) that the characteristics are given by

$$y = -e^{-x} + \xi, \; y = e^{-x} + \eta, \text{ for } \{\xi, \eta\} \in \mathbb{R}$$
$$\Rightarrow \xi = y + e^{-x}, \; \eta = y - e^{-x} \tag{15}$$

From the characteristics we have

$$\frac{d\xi}{dx} = -e^{-x}, \; \frac{d\xi}{dy} = 1$$
$$\frac{d\eta}{dx} = e^{-x}, \; \frac{d\eta}{dy} = 1$$

We can apply these results to evaluate the derivatives of $u(x, y)$ using the chain rule.

First, consider u_x.

$$u_x = \frac{d}{dx}U(\xi, \eta) = U_\xi \frac{d\xi}{dx} + U_\eta \frac{d\eta}{dx} = e^{-x}\left(U_\eta - U_\xi\right)$$

Differentiating this result with respect to x gives

$$u_{xx} = \frac{d^2}{dx^2}U(\xi, \eta) = -e^{-x}\left(U_\eta - U_\xi\right) + e^{-x}\left(U_{\eta\xi}\frac{d\xi}{dx} + U_{\eta\eta}\frac{d\eta}{dx} - U_{\xi\xi}\frac{d\xi}{dx} - U_{\xi\eta}\frac{d\eta}{dx}\right)$$
$$\Rightarrow u_{xx} = e^{-x}\left(U_\xi - U_\eta\right) + e^{-2x}\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right)$$

Next, we can evaluate the y derivatives.

$$u_y = \frac{d}{dy}U(\xi, \eta) = U_\xi \frac{d\xi}{dy} + U_\eta \frac{d\eta}{dy} = U_\xi + U_\eta$$

Differentiating this result with respect to y gives

$$u_{yy} = \frac{d^2}{dy^2}U(\xi, \eta) = U_{\xi\xi}\frac{d\xi}{dy} + U_{\xi\eta}\frac{d\eta}{dy} + U_{\eta\xi}\frac{d\xi}{dy} + U_{\eta\eta}\frac{d\eta}{dy} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting into the original PDE, we obtain the following equation for $U(\xi, \eta)$.

$$e^{2x}u_{xx} - u_{yy} = e^{2x}\left[e^{-x}\left(U_\xi - U_\eta\right) + e^{-2x}\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right)\right] - U_{\xi\xi} - 2U_{\xi\eta} - U_{\eta\eta} = 0$$

Simplifying, we obtain

$$e^{x}\left(U_\xi - U_\eta\right) = 4U_{\xi\eta}$$

We can solve for e^x using the characteristics given in Equation 15.

$$\xi - \eta = 2e^{-x} \Rightarrow e^{-x} = \frac{\xi - \eta}{2} \Rightarrow e^{x} = \frac{2}{\xi - \eta}$$

Substituting this result into the PDE, we obtain the following equation for $U(\xi, \eta)$

$$\boxed{2(\xi - \eta)U_{\xi\eta} = U_{\xi} - U_{\eta}} \tag{16}$$

Problem 4

The equation in Problem 1(b), $u_{xx} + 2u_{xy} + u_{yy} = 0$ has initial conditions $u(x, x) = x^2$, $u_n(x, x) = 0$, for $-\infty < x < \infty$. Does this IVP have a solution? Explain your answer.

Similar to a first-order PDE, if the characteristics of a linear second-order PDE are anywhere tangent to a characteristic, then we expect no solution in general (see class notes on 2/24/06). As a result, we begin our analysis by discussing the characteristic coordinates found in Problem 1(b).

Recall that this is an elliptic PDE with a single set of characteristics given by $y = x + \eta$ for $\eta \in \mathbb{R}$. Notice that the initial curve Γ is given by the line $y = x$. As a result, Γ is a characteristic (i.e., the one indexed by $\eta = 0$). Recall from Problem 4 in [1], when Γ is coincident with a characteristic there are only two outcomes: (1) either there is no solution to the IVP or (2) there are infinitely many solutions – depending on the initial value prescribed on Γ. As a result, by examining the characteristics alone, we have a hint that there could be no solution to this problem!

To proceed, we'll follow the derivation presented in class on 2/24/06. First, note that the initial value curve Γ (given by $y = x$) can be parameterized as follows.

$$x_0(\eta) = \eta, \; y_0(\eta) = \eta$$

The normal vector $\mathbf{n}(\eta)$ to Γ is given by

$$\mathbf{n}(\eta) = \begin{bmatrix} \dot{y}_0(\eta) \\ -\dot{x}_0(\eta) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, the gradient of u is given by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

As a result, the normal derivative (denoted $\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u$) is given by

$$u_x - u_y = u_n(x, x) = 0 \tag{17}$$

Since this condition applies everywhere along Γ we can substitute $u(x, x) = x^2$.

$$u_x(x, x) - u_y(x, x) = 2x = 0$$

As a result, we find that the initial conditions, $u(x,x) = x^2$ and $u_n(x,x) = 0$, are only consistent for $(x,y) = (0,0)$ (i.e., at the origin). As a result, **no solution exists to this IVP**.

As an alternate proof, we can evaluate the condition presented in class on 2/24/06 that must hold for a solution to a second-order linear PDE initial value problem to exist.

$$\begin{bmatrix} a & b & c \\ \dot{x}_0 & \dot{y}_0 & 0 \\ 0 & \dot{x}_0 & \dot{y}_0 \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} \nu_0 \\ \dot{p}_0 \\ \dot{q}_0 \end{bmatrix} \tag{18}$$

Recall that this expression encapsulates all of the constraints on the solution u. In addition, for a solution to exist, we showed that

$$\det \begin{bmatrix} a & b & c \\ \dot{x}_0 & \dot{y}_0 & 0 \\ 0 & \dot{x}_0 & \dot{y}_0 \end{bmatrix} \neq 0 \Rightarrow a\dot{y}_0^2 - b\dot{x}_0\dot{y}_0 + c\dot{x}_0^2 \neq 0$$

That is, the constraint equations must be linearly independent. For this problem, $\{a = 1, b = 2, c = 1\}$ and, as shown previously, $\dot{x}_0 = \dot{y}_0 = 1$. As a result, we find

$$a\dot{y}_0^2 - b\dot{x}_0\dot{y}_0 + c\dot{x}_0^2 = 1 - 2 + 1 = 0$$

In conclusion, the constraint equations given by Equation 18 are not linearly independent and, as a result, no solution exists to this IVP.

Problem 5

Consider the equation $u_{xx} - u_{yy} + u_x + u_y = 0$. Using characteristics $\xi = x + y$, $\eta = x - y$, **derive** the general solution to the PDE:

$$\begin{aligned} U(\xi,\eta) &= e^{-\eta/2} g(\xi) + h(\eta) \\ \Rightarrow u(x,y) &= e^{-(x-y)/2} g(x+y) + h(x-y) \end{aligned}$$

Here $g(\xi)$ and $h(\eta)$ are arbitrary functions. Verify that this general solution satisfies the PDE.

Notice that the characteristic coordinates can be inverted as follows.

$$\bar{x} = \frac{\xi + \eta}{2}, \ \bar{y} = \frac{\xi - \eta}{2}$$

As a result, we can express the general solution in (ξ,η)-coordinates as $u(\bar{x},\bar{y}) = U(\xi,\eta)$. As was done in Problem 2, we can evaluate the derivatives of $U(\xi,\eta)$ to obtain a PDE in (ξ,η)-coordinates.

To begin our analysis, note that the following derivatives can be directly obtained from the characteristics

$$\frac{d\xi}{dx} = 1, \ \frac{d\xi}{dy} = 1, \ \frac{d\eta}{dx} = 1, \ \frac{d\eta}{dy} = -1 \tag{19}$$

We can apply these expressions to evaluate u_x using the chain rule.

$$u_x = \frac{d}{dx}U(\xi,\eta) = U_\xi \frac{d\xi}{dx} + U_\eta \frac{d\eta}{dx} = U_\xi + U_\eta$$

Next, consider u_{xx}.

$$u_{xx} = \frac{d^2}{dx^2}U(\xi,\eta) = U_{\xi\xi}\frac{d\xi}{dx} + U_{\xi\eta}\frac{d\eta}{dx} + U_{\eta\xi}\frac{d\xi}{dx} + U_{\eta\eta}\frac{d\eta}{dx}$$

$$\Rightarrow u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Similarly, u_y is given by the following expression.

$$u_y = \frac{d}{dy}U(\xi,\eta) = U_\xi \frac{d\xi}{dy} + U_\eta \frac{d\eta}{dy} = U_\xi - U_\eta$$

Finally, we can evaluate u_{yy}.

$$u_{yy} = \frac{d^2}{dy^2}U(\xi,\eta) = \left(U_{\xi\xi}\frac{d\xi}{dy} + U_{\xi\eta}\frac{d\eta}{dy}\right) - \left(U_{\eta\xi}\frac{d\xi}{dy} + U_{\eta\eta}\frac{d\eta}{dy}\right)$$

$$\Rightarrow u_{yy} = U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

At this point we can substitute back into the original PDE to obtain a PDE in (ξ,η)-coordinates.

$$u_{xx} - u_{yy} + u_x + u_y = 0$$

$$\Rightarrow (U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}) - (U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}) + (U_\xi + U_\eta) + (U_\xi - U_\eta) = 0$$

Simplifying, we find

$$4U_{\xi\eta} + 2U_\xi = 0 \Rightarrow U_{\xi\eta} + \frac{1}{2}U_\xi = 0$$

This PDE for $U(\xi,\eta)$ can be solved using the method presented in class of 1/27/06.

$$U_{\xi\eta} = -\frac{1}{2}U_\xi \Rightarrow \frac{U_{\xi\eta}}{U_\xi} = -\frac{1}{2}$$

$$\Rightarrow \frac{d}{d\eta}\ln(U_\xi) = -\frac{1}{2}$$

Integrating both sides of this expression by η gives

$$\ln(U_\xi) = -\frac{1}{2}\eta + g(\xi)$$

where $g(\xi)$ is an arbitrary constant of integration (which in general can depend on the independent variable ξ). Exponentiating this expression yields the following result.

$$U_\xi = e^{-\eta/2+g(\xi)} = e^{-\eta/2}e^{g(\xi)} = e^{-\eta/2}g(\xi)$$

Notice that we have used the fact that $e^{g(\xi)}$ is an arbitrary function of ξ and, as a result, can be replaced notationally by the general function $g(\xi)$.

To complete our analysis, we can integrate by ξ as follows.

$$U(\xi,\eta) = e^{-\eta/2}\int_{-\infty}^{\xi} g(s)ds + h(\eta).$$

Since $\int_{-\infty}^{\xi} g(s)ds$ is an arbitrary function of ξ it can be replaced, once again, by $g(\xi)$. In conclusion, we have demonstrated the desired result. (QED)

$$\boxed{\begin{array}{c} U(\xi,\eta) = e^{-\eta/2}g(\xi) + h(\eta) \\ \Rightarrow u(x,y) = e^{-(x-y)/2}g(x+y) + h(x-y) \end{array}} \tag{20}$$

Before we verify the PDE in (x, y)-coordinates, let's verify it in the transformed (ξ, η)-coordinate system. Recall that, in this system, the PDE was given by $U_{\xi\eta} + \frac{1}{2}U_\xi = 0$. From Equation 20 we have

$$U_\xi = e^{-\eta/2}g'(\xi),\ U_{\xi\eta} = -\frac{1}{2}e^{-\eta/2}g'(\xi)$$

Substituting into the PDE we have

$$-\frac{1}{2}e^{-\eta/2}g'(\xi) + \frac{1}{2}e^{-\eta/2}g'(\xi) = 0$$

While the solution satisfies the PDE in the (ξ, η)-coordinate system, this is not a full proof. To compete our analysis, we will substitute the (x, y)-coordinate system solution from Equation 20 into the original PDE. To begin, let's compute the necessary derivatives.

$$u_x = -\frac{1}{2}e^{-(x-y)/2}g(x+y) + e^{-(x-y)/2}g'(x+y) + h'(x-y)$$

$$\Rightarrow u_x = e^{-(x-y)/2}\left[-\frac{1}{2}g(x+y) + g'(x+y)\right] + h'(x-y)$$

Taking the second derivative with respect to x yields

$$\begin{aligned} u_{xx} =& \frac{1}{4}e^{-(x-y)/2}g(x+y) - \frac{1}{2}e^{-(x-y)/2}g'(x+y) - \frac{1}{2}e^{-(x-y)/2}g'(x+y) + \ldots \\ & e^{-(x-y)/2}g''(x+y) + h''(x-y) \end{aligned}$$

$$\Rightarrow u_{xx} = e^{-(x-y)/2}\left[\frac{1}{4}g(x+y) - g'(x+y) + g''(x+y)\right] + h''(x-y)$$

Similarly, for the y derivatives, we find

$$u_y = \frac{1}{2}e^{-(x-y)/2}g(x+y) + e^{-(x-y)/2}g'(x+y) - h'(x-y)$$

$$\Rightarrow u_y = e^{-(x-y)/2}\left[\frac{1}{2}g(x+y) + g'(x+y)\right] - h'(x-y)$$

Taking the second derivative with respect to y yields

$$\begin{aligned} u_{yy} =& \frac{1}{4}e^{-(x-y)/2}g(x+y) + \frac{1}{2}e^{-(x-y)/2}g'(x+y) + \frac{1}{2}e^{-(x-y)/2}g'(x+y) + \ldots \\ & e^{-(x-y)/2}g''(x+y) + h''(x-y) \end{aligned}$$

$$\Rightarrow u_{yy} = e^{-(x-y)/2}\left[\frac{1}{4}g(x+y) + g'(x+y) + g''(x+y)\right] + h''(x-y)$$

At this point we can evaluate the PDE $u_{xx} - u_{yy} + u_x + u_y = 0$ directly, however for simplicity let's begin by considering the term $u_{xx} - u_{yy}$.

$$u_{xx} - u_{yy} = -2e^{-(x-y)/2}g'(x+y)$$

In addition, consider the term $u_x + u_y$.

$$u_x + u_y = 2e^{-(x-y)/2}g'(x+y)$$

In conclusion, we find the the general solution for $u(x, y)$ given by Equation 20 satisfies the PDE (i.e., $u_{xx} - u_{yy} + u_x + u_y = 0$). (QED).

References

[1] Douglas R. Lanman. Problem Set 3. `http://mesh.brown.edu/dlanman/courses/en202/HW3.pdf`.