EN 202: Problem Set 2

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Problem 1

Find $u(x, y)$ satisfying

$$u_x + u_y = u^2,\ u(x, -x) = x,\ -\infty \leq x \leq \infty$$

Sketch the characteristic curves and show the domain of validity of the solution in the (x, y)-plane.

In this problem we will examine the solution to a nonhomogeneous first-order PDE with constant coefficients. In order to solve this problem we can follow a typical approach: (1) find an explicit form for the characteristics, (2) transform the PDE to an ODE along these characteristics, and (3) conclude by determining the domain of validity of our solution.

Consider the vector field $\mathbf{v}$ defined as

$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi, \eta) \\ \bar{y}_\xi(\xi, \eta) \end{bmatrix} = \begin{bmatrix} a(x, y) \\ b(x, y) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, the gradient of u is given by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Note that the PDE can be restated as $\mathbf{v} \cdot \nabla u = u^2$. Once again, this equation represents a constraint on the component of the gradient of $u(x, y)$ which is parallel to $\mathbf{v}$. To proceed, we want to find a set of characteristics $\{\bar{x}(\xi, \eta), \bar{y}(\xi, \eta)\}$, with tracing parameter ξ, such that

$$\{\bar{x}_\xi = a(\bar{x}, \bar{y}) = 1, \bar{y}_\xi = b(\bar{x}, \bar{y}) = 1\} \tag{1}$$

Note that our initial curve Γ is the line $y = -x$ and can be parameterized as

$$\bar{x}(0, \eta) = x_0(\eta) = \eta$$

$$\bar{y}(0, \eta) = y_0(\eta) = -\eta$$

Integrating Equation 1 with respect to ξ gives $\bar{x}(\xi, \eta) = \xi + \phi(\eta)$ and $\bar{y}(\xi, \eta) = \xi + \psi(\eta)$. Recall that, along our initial curve Γ, $\xi = 0$; as a result, $\phi(\eta) = \eta$ and $\psi(\eta) = -\eta$. In conclusion, the characteristics are given by

$$\bar{x}(\xi, \eta) = \xi + \eta$$

$$\bar{y}(\xi, \eta) = \xi - \eta$$

We can invert these coordinates to obtain closed-form expressions for (ξ, η) as follows.

$$\bar{\xi}(x, y) = \frac{x + y}{2} \tag{2}$$

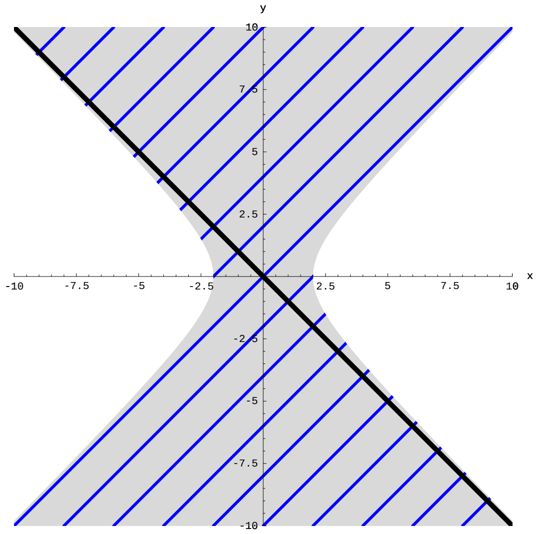


Figure 1: Sketch of characteristics (blue) and domain of validity (gray) found in Problem 1.

$$\bar{\eta}(x,y) = \frac{x-y}{2} \tag{3}$$

At this point, we turn our attention to the solution of the PDE. Notice that, by application of the chain rule, we have

$$U(\xi,\eta) = u(\bar{x},\bar{y})$$
$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = u_x + u_y = u^2.$$
$$\Rightarrow U_\xi(\xi,\eta) = U^2(\xi,\eta)$$

We can solve this ODE using standard methods as follows.

$$\frac{U_\xi}{U^2} = 1 \Rightarrow U^{-2}U_\xi = 1$$

$$\Rightarrow -\frac{d}{d\xi}U^{-1} = \frac{d}{d\xi}\left(\xi + \phi(\eta)\right)$$

$$\Rightarrow U(\xi,\eta) = \frac{-1}{\xi + \phi(\eta)}$$

Recall that on Γ, $U(x_0(\eta), y_0(\eta)) = u_0(\eta) = \eta$ and $\xi = 0$. As a result, we find that $\phi(\eta) = -1/\eta$. Substituting into the previous result, we find

$$U(\xi,\eta) = \frac{\eta}{1-\xi\eta}$$

Substituting the results from Equations 2 and 3, we find the solution for $u(x,y)$ is given by

$$\boxed{u(x,y) = \frac{2(x-y)}{y^2 - x^2 + 4} \text{ for } -\sqrt{y^2+4} < x < \sqrt{y^2+4} \text{ and } -\infty \leq y \leq \infty} \tag{4}$$

Notice that the denominator of the solution goes to zero for $y = \pm\sqrt{x^2-4}$. At this point the solution is undefined. As a result, our domain of validity $\mathcal{D}$ is given by $-\sqrt{y^2+4} <$

$x < \sqrt{y^2+4}$ and $-\infty \leq y \leq \infty$. The domain of validity is shaded gray in Figure 1. In addition, the characteristic curves given by Equation 3 (for several values of η) are shown in blue.

Problem 2

Find the solution to the PDE with the initial conditions

$$e^{-x}u_x + u_y = 0,\ y \geq 0$$

$$u(x,0) = u_0(x),\ -\infty \leq x \leq \infty$$

Give the domain of definition of the solution. Illustrate the solution with the appropriate graphs for the case in which $u_0(x) = e^{-\frac{1}{2}x^2}$.

Note: This problem is very similar to Problem 3 in Problem Set 1, with $a(x,y) = 1 + kx$ now equal to $a(x,y) = e^{-x}$. As a result, this solution will follow a similar derivation as [1].

As discussed in class on 2/3/06, this is a homogeneous first-order PDE with nonconstant coefficients of the form $au_x + bu_y + cu = f$, where $\{a, b, c, f\}$ are functions of x and y. We want to solve the Cauchy problem, in which the value of $u(x,y)$ is prescribed on some curve Γ in $\mathbb{R}$ (Note that, in this case, Γ is the line $y = 0$). As discussed in class on 1/30/06, the solution to this initial value problem will be valid in some domain $\mathcal{D}$ (possibly empty) of $\mathbb{R}$ that contains Γ. To begin, let's define the vector field $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} a(x,y) \\ b(x,y) \end{bmatrix} = \begin{bmatrix} e^{-x} \\ 1 \end{bmatrix}$$

Similarly, the gradient of u is given by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

As before, we can restate the PDE as $\mathbf{v} \cdot \nabla u = 0$. Once again, this equation represents a constraint on the component of the gradient of $u(x,y)$ which is parallel to $\mathbf{v}$. To proceed, we want to find a set of curves (i.e. characteristics) $\{\bar{x}(\xi,\eta), \bar{y}(\xi,\eta)\}$, with tracing parameter ξ, such that

$$\left\{\bar{x}_\xi = a(\bar{x},\bar{y}) = e^{-\bar{x}}, \bar{y}_\xi = b(\bar{x},\bar{y}) = 1\right\} \tag{5}$$

Note that, for any fixed η, the curve traced by ξ is everywhere tangent to the vector field $\mathbf{v}$, by construction. At this point, we want to modify our characteristics such that $\xi = 0$ corresponds to the intersection of the characteristic with the initial value curve Γ. That can be done by choosing

$$\bar{x}(0,\eta) = x_0(\eta) = \eta,\ \text{for } -\infty \leq \eta \leq \infty$$

$$\bar{y}(0,\eta) = y_0(\eta) = 0$$

Before we determine the explicit form of the characteristics, let's solve for the general form of u. Applying the chain rule we find

$$U(\xi,\eta) = u(\bar{x},\bar{y})$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = e^{-x}u_x + u_y = 0$$

Integrating with respect to ξ, we obtain $U(\xi,\eta) = \phi(\eta)$. At this point, we can apply the initial condition on Γ to solve for $\phi(\eta)$. Since $U(x_0(\eta), y_0(\eta)) = u_0(\eta)$, we find the general solution to the PDE is given by

$$U(\xi,\eta) = u_0(\eta) \text{ for } -\infty \leq \eta \leq \infty \tag{6}$$

To complete our analysis, we must solve Equation 5 to obtain a closed-form expression for the characteristics. Let's begin by integrating $\bar{y}_\xi = 1$ to obtain

$$\bar{y}(\xi,\eta) = \int_0^\xi dz + \phi(\eta) = \xi + \phi(\eta)$$

If we apply the initial condition $\bar{y}(0,\eta) = 0$ we find $\phi(\eta) = 0$. In conclusion, the $\bar{y}$-component of the characteristics is given by

$$\bar{y}(\xi,\eta) = \xi$$

To determine $\bar{x}(\xi,\eta)$ we can solve the ODE $\bar{x}_\xi = e^{-\bar{x}} \Rightarrow \log(\bar{x}_\xi) = \bar{x}$ given in Equation 5. The general solution to this ODE is given by $\bar{x}(\xi,\eta) = \log(\xi + \phi(\eta))$ (Note: this is verified in the attached *Mathematica* notebook). Substituting the initial condition $\bar{x}(0,\eta) = x_0(\eta) = \eta$ we determine that $\phi(\eta) = e^\eta$. In conclusion, the characteristics are given by

$$\bar{x}(\xi,\eta) = \log(\xi + e^\eta) \tag{7}$$

$$\bar{y}(\xi,\eta) = \xi \tag{8}$$

where log is the natural logarithm. Solving this system of equations for η, we obtain

$$\bar{x} = \log(\bar{y} + e^\eta) \Rightarrow e^\eta = e^{\bar{x}} - \bar{y}$$

$$\eta(x,y) = \log\left(e^x - y\right)$$

To find the general solution for $u(x,y)$, we can substitute for $\eta(x,y)$ in Equation 6.

$$u(x,y) = u_0(\log\left(e^x - y\right)), \text{ for } -\infty \leq \log\left(e^x - y\right) \leq \infty$$

To determine the domain $\mathcal{D}$ of validity, we can take the exponential of the inequality to obtain

$$e^x - y \geq 0 \Rightarrow y \leq e^x \Rightarrow x \geq \log(y)$$

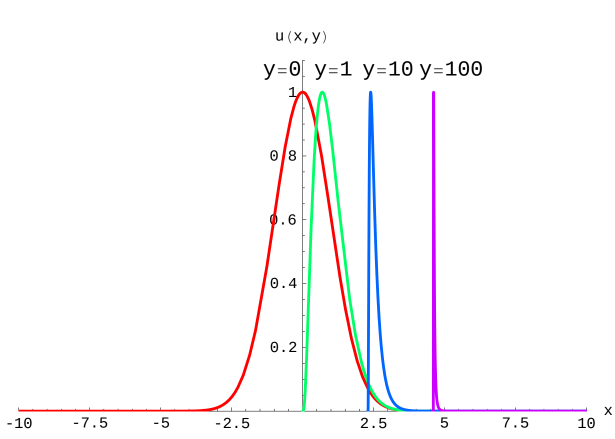
For $y \geq 0$ this yields the general solution:

$$\boxed{u(x,y) = u_0(\log\left(e^x - y\right)) \text{ for } x \geq \log(y) \text{ and } y \geq 0} \tag{9}$$

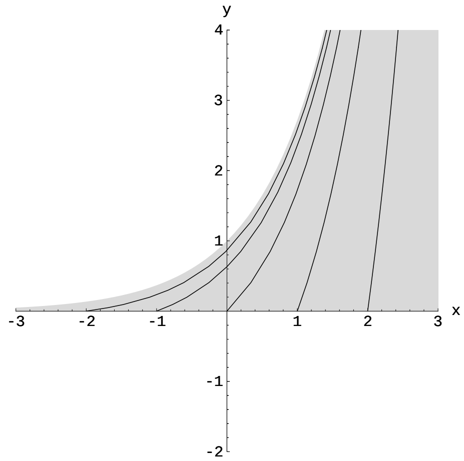
For the specific case $u_0(x) = e^{-\frac{1}{2}x^2}$, we find the solution

$$\boxed{u(x,y) = e^{-\frac{1}{2}[\log(e^x - y)]^2} \text{ for } x \geq \log(y) \text{ and } y \geq 0} \tag{10}$$

Plots of this specific solution were generated in *Mathematica* and are shown in Figure 2(a). The *Mathematica* notebook is included at the end of this write-up. Note that, in the figures, the cases $y = \{0, 1, 10, 100\}$ are shown in {red, green, blue, purple}, respectively. Similarly, the domain $\mathcal{D}$ of validity of the solution, given by the region $x \geq \log(y)$ and $y \geq 0$, is shaded gray in Figure 2(b) (solid lines represent characteristics evaluated for various values of η).



(a) illustration of $u(x, y)$ for $u_0(x) = e^{-\frac{1}{2}x^2}$



(b) domain of validity $\mathcal{D}$ for $u(x, y)$

Figure 2: Illustration of $u(x, y)$ and its domain of validity found in Problem 2.

Problem 3

Consider the initial value problem $u_x + xu_y = 0$, $u(x, 0) = u_0(x)$ for $-\infty \leq x \leq \infty$ with $u_0(x)$ given in Parts (a) and (b) below. To answer the questions, see if the initial curve is tangent to a characteristic. If so, see if the initial data at the point of tangency is consistent with the PDE at the point.

(a) Explain why there are no solutions in $u_0(x) = \sin(x)$.
(b) Is there a solution if $u_0(x) = \cos(x)$? If so, find it!

Part (a)

In order to prove the nonexistence of solutions to the PDE, we will follow the method presented in class on 2/6/06. Let us begin by finding the normal vector $\mathbf{w}$ for the initial curve Γ. In this example, the initial curve is the line $y = 0$ which can be parameterized as follows.

$$\bar{x}(\xi, \eta) = x_0(\eta) = \eta$$

$$\bar{y}(\xi, \eta) = y_0(\eta) = 0$$

As a result, the normal vector $\mathbf{w}$ for the initial curve Γ is given by

$$\mathbf{w} = \begin{bmatrix} y_0'(\eta) \\ -x_0'(\eta) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which is simply the negative $\hat{y}$-axis. Recall from class on 2/1/06 that a necessary condition for the (ξ, η)-coordinates to be invertible (i.e., transformed to (x, y)-coordinates) is that the Jacobian determinate $\Delta(\xi, \eta) \neq 0$ on Γ. Further recall that, on Γ, the Jacobian determinate will be zero anywhere the initial curve Γ is tangent to a characteristic. If we express the vector field in the typical manner as

$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi,\eta) \\ \bar{y}_\xi(\xi,\eta) \end{bmatrix} = \begin{bmatrix} a(x,y) \\ b(x,y) \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

then we can express the condition that Γ is tangent to a characteristic as

$$\mathbf{v} \cdot \mathbf{w} = 0 \Rightarrow \Delta(\xi,\eta) = 0 \text{ on } \Gamma$$

Substituting for $\mathbf{v}$ and $\mathbf{w}$ we find

$$\begin{bmatrix} 1 & x \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -x$$

In conclusion, we find that the initial curve Γ will be tangent to a characteristic only at $x = 0$. In general, we expect that the PDE will not have a solution at this point. We can verify this claim by evaluating the PDE at $(x,y) = (0,0)$.

$$u_x(x,y) + xu_y(x,y) = 0 \Rightarrow u_x(0,0) = 0 \tag{11}$$

But if we examine the initial data $u(x,0) = u_0(x) = \sin(x)$, then we have an additional constraint that $u_x(x,0) = \cos(x)$. Evaluating this condition for $(x,y) = (0,0)$ we find

$$u_x(0,0) = \cos(0) = 1 \tag{12}$$

In conclusion, we find that Equations 11 and 12 are inconsistent (i.e., $0 \neq 1$). As a result, the initial data $u_0(x) = \sin(x)$ is inconsistent with the requirements of the PDE on the initial curve Γ and, as a result, there can be no solution of the initial value problem as posed. QED

Part (b)

First, let's verify that the initial data $u_0(x) = \cos(x)$ is consistent with the requirements of the PDE given by Equation 11. Taking the derivative of $u_0(x) = \cos(x)$ with respect to x gives $u_x(x,0) = -\sin(x)$ which implies that $u_x(0,0) = -\sin(0) = 0$. Unlike in Part (a), it appears that the initial data is consistent with the requirements of the PDE (despite the point of tangency at $x = 0$). We'll proceed as usual and examine the behavior of the solution and its domain of validity.

As usual, we will begin by seeking a closed-form solution for the characteristics such that

$$\{\bar{x}_\xi = a(\bar{x},\bar{y}) = 1, \bar{y}_\xi = b(\bar{x},\bar{y}) = \bar{x}\} \tag{13}$$

Since our initial value curve Γ is the line $y = 0$ we can parameterize it as follows

$$\bar{x}(0,\eta) = x_0(\eta) = \eta, \text{ for } -\infty \leq \eta \leq \infty$$

$$\bar{y}(0,\eta) = y_0(\eta) = 0$$

Before we determine the explicit form of the characteristics, let's solve for $U(\xi,\eta)$. Applying the chain rule we find

$$U(\xi,\eta) = u(\bar{x},\bar{y})$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial \bar{x}}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial \bar{y}}{\partial \xi} = u_x + xu_y = 0$$

Integrating with respect to ξ, we obtain $U(\xi,\eta) = \phi(\eta)$. At this point, we can apply the initial condition on Γ to solve for $\phi(\eta)$. Since $U(x_0(\eta), y_0(\eta)) = u_0(\eta) = \cos(\eta)$, we find the general solution to the PDE is given by

$$U(\xi,\eta) = \cos(\eta) \text{ for } -\infty \leq \eta \leq \infty \tag{14}$$

To complete our analysis, we must solve Equation 13 to obtain a closed-form expression for the characteristics. Let's begin by integrating $\bar{x}_\xi = 1$ to obtain

$$\bar{x}(\xi,\eta) = \int_0^\xi dz + \phi(\eta) = \xi + \phi(\eta)$$

If we apply the initial condition $\bar{x}(0,\eta) = \eta$ we find $\phi(\eta) = \eta$. In conclusion, the $\bar{x}$-component of the characteristics is given by

$$\bar{x}(\xi,\eta) = \xi + \eta$$

Since $\bar{y}_\xi = \bar{x}$ we can substitute the previous result to find $\bar{y}_\xi = \xi + \eta$. Integrating both sides with respect to ξ yields the following result.

$$\bar{y}(\xi,\eta) = \int_0^\xi z dz + \eta \int_0^\xi dz + \phi(\eta) = \frac{1}{2}\xi^2 + \xi\eta + \phi(\eta)$$

Applying the initial condition that $\bar{y}(0,\eta) = y_0(\eta) = 0$ gives $\phi(\eta) = 0$. In conclusion, the characteristics are given by

$$\bar{x}(\xi,\eta) = \xi + \eta \tag{15}$$

$$\bar{y}(\xi,\eta) = \frac{1}{2}\xi^2 + \xi\eta \tag{16}$$

Substituting $\xi = x - \eta$ into Equation 16 gives the following solution for η.

$$y = \frac{1}{2}(x-\eta)^2 + (x-\eta)\eta \Rightarrow \eta^2 = x^2 - 2y$$

$$\eta(x,y) = \pm\sqrt{x^2 - 2y}$$

As shown in Figure 3 the characteristics given by $y = \frac{1}{2}x^2 - \frac{1}{2}\eta$ are a family of parabolas. For $\eta \neq 0$, the characteristics intersect the initial curve Γ twice. This situation is similar to that shown in class on 2/6/06 and corresponds to the positive and negative roots of $\eta(x,y)$. Since $u_0(x) = u_0(-x)$ for the even function $u_0(x) = \cos(x)$ the solution exists and can be expressed by substituting for η in Equation 14. In conclusion, the solution to this Cauchy problem is given by

$$\boxed{u(x,y) = \cos(\sqrt{x^2 - 2y}) \text{ for } \{x \geq \sqrt{2y} \text{ and } x \leq -\sqrt{2y}, y \geq 0\} \text{ and } \{-\infty \leq x \leq \infty, y < 0\}}$$

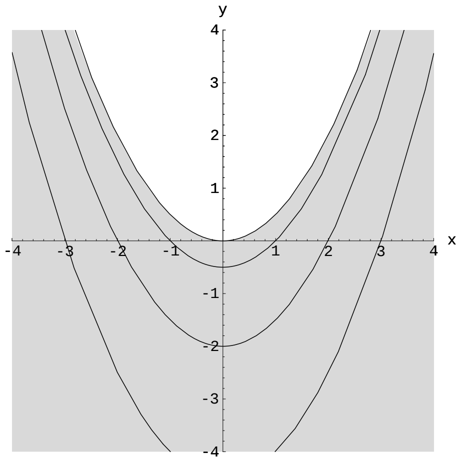


Figure 3: Domain of validity $\mathcal{D}$ and characteristics found in Problem 3(b).

Problem 4

Solve $uu_x + u_y = 0$, $u(x, 0) = \sin(x)$. Find the value of the time T for which the waves break. That is, find T for which $u(x, y)$ is well defined for $0 \leq y \leq T$ but $u(x, T)$ has a vertical tangent at some point(s). Find these location(s) of the breaking of the waves.

As discussed in class on 2/8/06, this problem focuses on the solution of a quasilinear first-order PDE. Notice that our initial curve Γ is the line $y = 0$. As a result, we can parameterize it as follows.

$$\bar{x}(\xi, \eta) = x_0(\eta) = \eta$$

$$\bar{y}(\xi, \eta) = y_0(\eta) = 0$$

In addition, we define the vector field $\mathbf{v}$ as

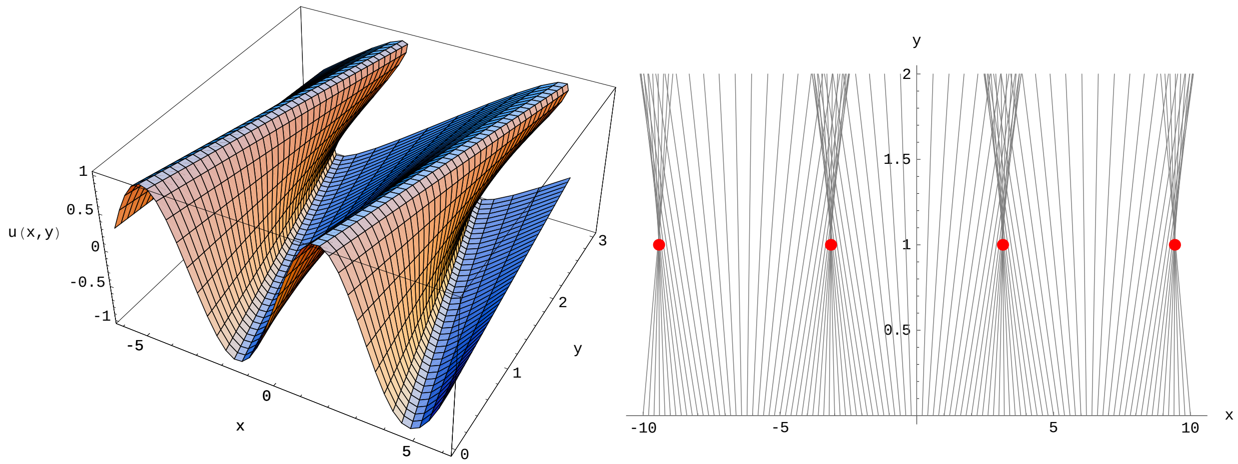
$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi, \eta) \\ \bar{y}_\xi(\xi, \eta) \end{bmatrix} = \begin{bmatrix} U(\xi, \eta) \\ 1 \end{bmatrix}$$

As usual, the PDE becomes an ODE along the characteristics such that $\mathbf{v} \cdot \nabla u = 0$. We can integrate with respect to ξ to determine $U(\xi, \eta) = \phi(\eta)$. Notice that, along Γ, $U(x_0(\eta), y_0(\eta)) = u_0(\eta) = \sin(\eta)$. Substituting into the previous result, we have

$$U(\xi, \eta) = \sin(\eta) \tag{17}$$

$$\mathbf{v} = \begin{bmatrix} \bar{x}_\xi(\xi, \eta) \\ \bar{y}_\xi(\xi, \eta) \end{bmatrix} = \begin{bmatrix} \sin(\eta) \\ 1 \end{bmatrix} \tag{18}$$

At this point, we require an explicit form of the characteristics. First, consider the solution to the ODE $\bar{y}_\xi(\xi, \eta) = 1$. Integrating with respect to ξ, we find $\bar{y} = \xi + \phi(\eta)$. Applying the initial value on Γ, where $y_0(\eta) = 0$ and $\xi = 0$, we find $\phi(\eta) = 0$. Next, consider the solution to the ODE $\bar{x}_\xi(\xi, \eta) = \sin(\eta)$. Integrating with respect to ξ, we find $\bar{x} = \xi \sin(\eta) + \psi(\eta)$. Using the initial value on Γ, where $x_0(\eta) = \eta$ and $\xi = 0$, we find $\psi(\eta) = \eta$. Combining these



(a) $u(x, y)$ solution surface (b) wave-breaking points and characteristics

Figure 4: Illustration of $u(x, y)$ and wave-break phenomena for Problem 4.

results with Equation 17, we obtain the following solution to the PDE given by a parametric surface in the (x, y, u)-space, with tracing parameters (ξ, η).

$$\boxed{\begin{aligned} \bar{x}(\xi,\eta) &= \xi \sin(\eta) + \eta \\ \bar{y}(\xi,\eta) &= \xi \\ U(\xi,\eta) &= \sin(\eta) \end{aligned}} \tag{19}$$

Although Equation 19 is a complete solution to the PDE, we'd like to find a closed-form solution for $u(x, y)$ if one exists. Combining the previous results, we find the following relation.

$$x = y \sin(\eta) + \eta \tag{20}$$

In general, this equation does not admit a straightforward solution for $\eta(x, y)$. As a result, we must be content with Equation 19 as the solution of the PDE. The parametric solution surface in the (x, y, z)-space is illustrated in Figure 4(a).

Although we cannot solve for $u(x, y)$ analytically, we can determine some of the properties of the solution surface, such as the location(s) of wave-breaking point(s). Recall that $u_x \to \infty$ when $u(x, y)$ has a vertical tangent; as discussed in class on 2/10/06, the presence of a vertical tangent occurs at a wave-breaking point – where the solution $u(x, y)$ becomes multivalued. Let us begin by differentiating Equation 20 with respect to x.

$$\frac{d}{dx}x = \frac{d}{dx}\left(y \sin(\eta) + \eta\right)$$
$$\Rightarrow 1 = y \cos(\eta)\eta_x + \eta_x$$
$$\Rightarrow \eta_x = \frac{1}{1 + y\cos(\eta)} \to \infty$$

In addition, we can differentiate the solution $U(\xi, \eta) = \sin(\eta)$ with respect to x.

$$\frac{d}{dx}\sin(\eta) = \cos(\eta)\eta_x = \frac{cos(\eta)}{1 + y\cos(\eta)} \to \infty$$

In general, $n_x \to \infty$ when the denominator of this equation tends to zero. As a result, we find the following constraint of the value(s) of η at the wave-breaking point(s).

$$1 + y\cos(\eta) = 0 \Rightarrow cos(\eta) = \frac{-1}{y} \tag{21}$$

We can take the square of this relation to obtain the following constraint on $\sin(\eta)$.

$$\cos^2(\eta) = 1 - \sin^2(\eta) = \frac{1}{y^2}$$

$$\Rightarrow sin(\eta) = \pm\frac{\sqrt{y^2-1}}{y}$$

Substituting this result into Equation 20, we find

$$x = \pm y\frac{\sqrt{y^2-1}}{y} + \eta = \pm\sqrt{y^2-1} + \eta \text{ for } \eta_x \to \infty$$

As a result, we conclude that at the wave-breaking point(s) we have

$$\eta = x \pm \sqrt{y^2-1} \tag{22}$$

In general, Equation 22 will have no solutions (and no vertical tangents) for $0 \leq y < 1$. For $y > 1$, Equation 22 will have two solutions (this corresponds to the state of the wave *after* breaking). Finally, for $y = 1$, Equation 22 has one unique solution which corresponds to the point(s) at which the wave breaks. As a result, the time T when the wave breaks is $y = 1$. To complete our analysis, we can substitute $y = 1$ into Equation 22 to find $\eta = x$ at the wave-breaking point(s). Substituting $\eta = x$ and $y = 1$ into Equation 21, we find that $\cos(x) = -1$ at the wave-breaking point(s); as a result, the wave-breaking points are given by

$$\boxed{\begin{array}{l} \text{wave-breaking time: } T = 1 \\ \text{wave-breaking points: } \{x = \pi(2k+1), k \in \mathbb{Z}, y = 1\} \end{array}} \tag{23}$$

The characteristics given by Equation 19 are sketched in Figure 4(b). Notice that the characteristics intersect at the wave-breaking points (indicated by red dots). This is consistent with our interpretation that the wave-breaking points are positions where $u(x, y)$ becomes multivalued.

References

[1] Douglas R. Lanman. Problem Set 1. `http://mesh.brown.edu/dlanman/courses/en202/HW1.pdf`.