# AM 255: Problem Set 5

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## Problem 1

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Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{j=0}^{4} a_j \frac{\partial^j u}{\partial x^j}. \tag{1}$$

Derive the condition for well-posedness. Is the problem always well posed if $\mathrm{Re}(a_4) < 0$?

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We begin our analysis by rewriting Equation 1 for $t \geq t_0$ as

$$u_t(x,t) = a_4 u_{xxxx}(x,t) + a_3 u_{xxx}(x,t) + a_2 u_{xx}(x,t) + a_1 u_x(x,t) + a_0 u(x,t), \tag{2}$$

with $2\pi$-periodic initial data

$$u(x,t_0) = f(x).$$

As done in previous problem sets, we proceed by assuming $f(x)$ is composed of a single wave

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{i\omega x} \hat{f}(\omega).$$

As a result, we have the following simple wave solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} e^{i\omega x} \hat{u}(\omega,t), \quad \hat{u}(\omega,0) = \hat{f}(\omega). \tag{3}$$

Equation 3 leads to the following solutions for the partial derivatives of $u(x,t)$.

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}} e^{i\omega x} \hat{u}_t(\omega,t) \tag{4}$$

$$u_x(x,t) = \frac{i\omega}{\sqrt{2\pi}} e^{i\omega x} \hat{u}(\omega,t) \qquad u_{xx}(x,t) = \frac{-\omega^2}{\sqrt{2\pi}} e^{i\omega x} \hat{u}(\omega,t) \tag{5}$$

$$u_{xxx}(x,t) = \frac{-i\omega^3}{\sqrt{2\pi}} e^{i\omega x} \hat{u}(\omega,t) \qquad u_{xxxx}(x,t) = \frac{\omega^4}{\sqrt{2\pi}} e^{i\omega x} \hat{u}(\omega,t) \tag{6}$$

Substituting Equations 3 through 6 into Equation 2 gives the following expression.

$$\frac{1}{\sqrt{2\pi}} e^{i\omega x} \hat{u}_t(\omega,t) = \frac{1}{\sqrt{2\pi}} e^{i\omega x} \left(a_4\omega^4 - i\omega^3 a_3 - a_2\omega^2 + i\omega a_1 + a_0\right) \hat{u}(\omega,t)$$

Canceling the common terms in $\frac{1}{\sqrt{2\pi}} e^{i\omega x}$ gives the following ordinary differential equation

$$\hat{u}_t(\omega,t) = \kappa \hat{u}(\omega,t), \quad \kappa := a_4\omega^4 - i\omega^3 a_3 - a_2\omega^2 + i\omega a_1 + a_0, \tag{7}$$

with the the general solution given by

$$\hat{u}(\omega,t) = e^{\kappa t}\hat{f}(\omega). \tag{8}$$

The $L_2$ norm of the general solution $u(x,t)$ can be written in the following form.

$$\|u(\cdot,t)\|^2 = \|e^{\kappa t}\hat{f}(\omega)\|^2 = |e^{\kappa t}|^2\,|\hat{f}(\omega)|^2 = e^{2\mathrm{Re}(\kappa)t}\,\|f(\cdot)\|^2$$
$$\Rightarrow \|u(\cdot,t)\| = e^{\mathrm{Re}(\kappa)t}\,\|f(\cdot)\| \tag{9}$$

Note that, in the proceeding equation, the following identities were applied: $|e^{\kappa t}|^2 = e^{2\mathrm{Re}(\kappa)t}$ and $|\hat{f}(\omega)|^2 = \|f(\cdot)\|^2$. The first expression naturally follows from the properties of the modulus of a complex exponential. The second identity corresponds to Parseval's relation.

At this point, we recall Definition 4.1.1. from page 110 in [1]. To briefly summarize, the general system of partial differential equations

$$u_t = P\left(x,t,\frac{\partial}{\partial x}\right)u, \quad t \ge t_0,$$

with initial data

$$u(x,t_0) = f(x),$$

will be well posed if, for every $t_0$ and every $f \in C^\infty(x)$: 1. there exists a unique solution $u(x,t) \in C^\infty(x,t)$, which is $2\pi$-periodic in every space dimension and 2. there are constants $\alpha$ and $K$, independent of $f$ and $t_0$, such that

$$\|u(\cdot,t)\| \le Ke^{\alpha(t-t_0)}\|f(\cdot)\|. \tag{10}$$

Comparing Equation 9 to Equation 10, it is apparent that Definition 4.1.1. will only hold for $K \le 1$ and

$$\boxed{\mathrm{Re}(\kappa) \le \alpha, \quad \kappa := a_4\omega^4 - i\omega^3 a_3 - a_2\omega^2 + i\omega a_1 + a_0,} \tag{11}$$

where $\alpha$ is a real constant. As a result, Equation 11 expresses the necessary condition for Equation 1 to be well posed. To complete our proof we must show that Equation 10 holds for any initial data

$$u(x,0) = f(x) = \frac{1}{\sqrt{2\pi}}\sum_{\omega=-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}.$$

Assuming Equation 11 holds, the general solution exists and is given by the following expression.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\sum_{\omega=-\infty}^{\infty} e^{\kappa(\omega)t+i\omega x}\hat{f}(\omega)$$

Applying Parseval's relation we find that the $L_2$ norm satisfies the following inequality.

$$\|u(\cdot,t)\|^2 = \sum_{\omega=-\infty}^{\infty} e^{2\mathrm{Re}(\kappa(\omega))t}\,|\hat{f}(\omega)|^2 \le e^{2\alpha t}\,\|f(\cdot)\|^2$$

In conclusion, Equation 10 holds for any initial data $u(x,0)$ if and only if the condition in Equation 11 is satisfied.

To complete our analysis we observe that **the problem is always well posed if $\mathbf{Re}(\mathbf{a_4}) < \mathbf{0}$**. This can be shown by examining the behavior of Equation 11 for $a_r = \text{Re}(a_4) < 0$. In this situation we have

$$\begin{aligned}\text{Re}(\kappa) &= \text{Re}(a_4\omega^4 - i\omega^3 a_3 - a_2\omega^2 + i\omega a_1 + a_0) \\ &\leq a_4\omega^4 + |a_3||\omega^3| + |a_2|\omega^2 + |a_1||\omega| + |a_0| \\ &= -|a_r|\omega^4 + |a_3||\omega|^3 + |a_2|\omega^2 + |a_1||\omega| + |a_0|.\end{aligned}$$

In the limit of large omega the highest order term $-|a_r|\omega^4$ dominates and, because the coefficient of this term is negative, we can always satisfy the well-posedness condition given by Equation 11 such that

$$lim_{\omega\to\pm\infty}\text{Re}(\kappa) \leq \alpha.$$

In addition, for any finite $\omega$, we can always choose a value of $\alpha$ such that $\text{Re}(\kappa) \leq \alpha$. In conclusion, we find that the problem is always well-posed if $\text{Re}(a_4) < 0$.

# References

[1] Bertil Gustafsson, Heinz-Otto Kreiss, and Joseph Oliger. *Time Dependent Problems and Difference Methods*. John Wiley & Sons, 1995.