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The Development of Geometric Knowledge

Intuition about geometry has long been a topic of interest to man. In ancient Rome, Socrates, as told by Plato in *Meno*, led a slave to conclude that to double the area of a square, he should double each of its sides, and then cut diagonally from midpoint to midpoint of each side. For Socrates, geometric reasoning was non-obvious; however, new geometric knowledge could be gained through abstract reasoning. In the mid 1800's, Immanuel Kant proposed that knowledge came about because of *necessary force*—the world provides experiences, that to be understood require u the building of higher and higher levels of knowledge and understanding. Geometric knowledge develops because of necessary experience with geometric objects, such that understanding requires higher level knowledge. (Kant 1855) Hannah Silberstein & Elizabeth Spelke, through interviews with children, found that children passed basic reasoning tasks about Euclidean geometry at about six years of age. (Silberstein H. & Spelke, E. S. 1998) This represents Piaget's *concrete operational* stage, a time when children become capable of reasoning about abstract objects. (Piaget 1969) It is clear from research that different geometric facts are learned different times. To understand the time-course of this process of learning, the presence of sample geometric facts must be tested.

The knowledge, seemingly simple for adults, that a rectangle is composed of two identical right triangles falls into the category of geometric knowledge that is predicted to develop in children at some stage after the preoperational stage. Such knowledge requires knowledge about both rectangles and triangles, particularly the special class of right triangles, as well as general geometric knowledge of segmentation (the splitting of one object into two other objects) and similarity. Furthermore, there may exist different levels of this knowledge: a child may know that he can take two identical triangle tiles and place them together in such a way as to create a rectangle (physical intuition). However, he may be unable to perform the same operation purely in his mind (abstract geometric intuition). Or he may be able to construct a rectangle from its constituent triangles, but be unable to segment a rectangle into identical triangles (or vice versa). Complete knowledge of this geometric fact requires that all of the above reasoning cases be completed successfully.

To determine the time period for the development of this knowledge, children must be tested over time at a series of tasks, each presented in sequence once the prior one has been successfully completed:

Test 1: Given two identical right triangular tiles, can the child construct a rectangle? Given enough time, the child should stumble upon the correct solution merely by playing with the two tiles and placing them together in different orientations.

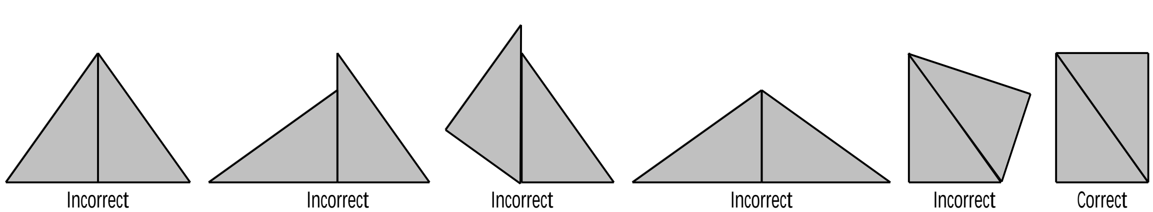

Figure 1

Possible outcomes for the placement of the triangles are shown in Figure 1. Once the child sees the proper solution, can he abstract it to right triangles in general? If the child succeeds in this task, with any triangles given to him, then he can be said to have passed the stage of physical intuition with respect to this aspect of geometric knowledge.

Test 2: Given a rectangle (drawn on a piece of paper), can a child properly create two identical right triangles by segmenting the rectangle? There are only two correct solutions to this problem: a line from the top left to the bottom right corner, or a line from the top right to the bottom left corner of the rectangle. This problem is more difficult than the previous one, since there are an infinite number of lines that can be drawn through the rectangle that would result in improper segmentation (there are only two non-transverse lines in this case) (Richards 1992).

Test 3: Given a right triangle on a piece of paper, can a child construct a rectangle that would be made of two such triangles properly placed together? This directly

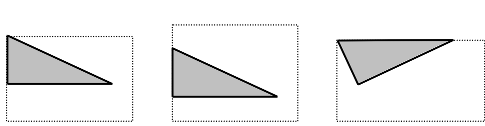

Figure 2

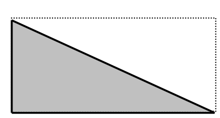

Figure 3

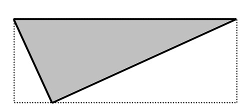

Figure 4

tests a child's ability to represent such geometric knowledge abstractly (using mental representations as opposed to physical objects). There are an infinite number of ways that a child could construct a rectangle given a triangle. Several are shown in Figure 2. There exists only one that properly duplicates and transforms the given triangle to create the expected rectangle.

This test is more complex than the ones above since there are fewer constraints on the possible answers a child can give; however, if a child does know that two identical right triangles, properly coupled, create a rectangle, then he should succeed in this task after only a few tries. It is vital that the child be tested using different triangle orientations. A child could falsely succeed in this task when given a particularly easy orientation (Figure 3) or falsely fail when given an orientation where the most easily constructed rectangle is not one that satisfies the constraints of the test (Figure 4).

One may ask: "What constitutes knowledge about a particular geometric fact?" To show knowledge about something, one must communicate that knowledge. This is particularly difficult for children with respect to more complex geometric facts since they often lack the vocabulary to describe their knowledge. They can, however, sufficiently communicate their knowledge through their success on certain mentally taxing tasks that require such knowledge. Test 3 above attempts to allow children to do just this. To successfully complete Test 3, a child must be able to represent the given triangle in his mind, and then be able to transform it, remembering its prior orientation. The child must then be capable of keeping the original and transformed triangle in mind while comparing the outline of the figure formed with the child's representation of a rectangle. From this description, the task seems almost impossible. Yet from about age twelve on, when children first come in contact with abstract geometry in school, such tasks are easily accomplished.

A child must also be capable of reasoning about objects using his knowledge. Test 2 above tries to test this capacity. A child who knows that a rectangle is made of two identical right triangles must use this knowledge to arrive at the conclusion that any rectangle can be segmented into such triangles. This task also requires mental representation, since the child must mentally represent the triangles that make up the given rectangle—he should not be allowed to draw out many lines to find the solution by trial and error. The child must reason that if he draws either of the two lines from corner to corner, he will create two triangles, and those triangles will be identical.

Children's performance on the above tests is expected to change over time, with young children failing all three tests. As a child grows, he may stumble upon the physical reality of the rule that two identical right triangles put together make a rectangle. However, prior to the concrete operational stage, he is expected only to pass the first of the three tests. He is still severely limited in his abilities to mentally represent concepts, and so should fail Tests 2 and 3. As the child approaches the formal operational stage, he should be able to abstract this physical knowledge, and should thus be capable of passing Tests 2 and 3.

This change in children's ability to succeed in the above tasks may be due to the development of two different capacities. Children may be unable initially to abstract the knowledge that they first learn physically into a form that is useful for mental tasks. By succeeding in the first Test, a child may show that he has learned the rule in a physical domain. However, his inability to perform on the second and third Tests may be due to his inability to translate that physical knowledge into an abstract form that he can then use to think about right triangles in general. If he are unable to abstract the rule, then, while he may be able to represent triangles and rectangles mentally (which has been shown to be within their capacities by around 6 years of age) they cannot apply the rule to these mental representations. (Silberstein, E. & Spelke, E.S.),

Alternatively, children may indeed be capable of abstracting the rule at around the same time that they are capable of representing mentally rectangles and triangles. They may instead be limited in the amount of information that they can maintain mentally at any given time. To succeed on the third Test, children must be able to keep the original triangle in memory, duplicate and transform it, keep the second triangle in memory, and then, using only the outline of the figure created by combining the two triangles, compare the new shape to a rectangle. This amounts to a large amount of information that must be maintained in memory for the child to succeed at the given task.

To distinguish between these two alternative explanations for a child's ability to succeed at the above tests, a further experiment must be performed. Ideally, such an experiment would test the child's ability to abstract knowledge versus his ability to hold items in memory for manipulation and comparison. Given a similarly complex geometric fact, such as the idea that three equal line segments when attached end to end to end form an equilateral triangle, do children succeed in moving such knowledge from the physical world to a mental world? If so, then clearly they are deficient in their ability to hold more complex objects in memory, since the level of complexity of the task is similar to the task of abstracting the fact that two identical right triangles form a rectangle. The complexity of the objects that must be kept in mind is far less—line segments have 180º symmetry and also have only 1 degree of freedom with regard to their size, which means that performing the mental transformations of the line segments is much less taxing on memory.

If, instead, the child fails this test, but instead passes one of similar complexity with regard to memory constraints, then the former explanation must be true. Given a triangle, can a child construct another similar triangle? This test draws on knowledge about the angles of a triangle and on the knowledge that two triangles are similar if their angles correspond. This task is as memory intensive as that in Test 3—the child must hold in memory the given triangle, and create a new triangle by duplicating and transforming it. He must then scale one of them, and compare them. Three objects must be maintained in memory: the given triangle, the newly constructed triangle, and a scaled copy of one of these to be compared with the other, each of which is as complex an object as those in Test 3. The complexity involved with abstracting such knowledge is less than that of Test 3: there is only a one-to-one correspondence between the similar triangles, whereas there is a two-to-one correspondence between the identical right triangles and the rectangle. Thus if children can succeed in this task, their deficiency lies in their ability to abstract complex geometric knowledge, and not in their ability to maintain objects in memory.

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