Assigned Reading:
- Eero Simoncelli “A Geometric View of Linear Algebra”
- Michael Jordan slightly more in depth linear algebra review
- Online Introductory Linear Algebra Book by Jim Hefferon.
  [http://joshua.smcvt.edu/linearalgebra/](http://joshua.smcvt.edu/linearalgebra/)

Notation
- Standard math textbook notation
  - Scalars are italic times roman: \( n, N \)
  - Vectors are bold lowercase: \( \mathbf{x} \)
    - Row vectors are denoted with a transpose: \( \mathbf{x}^T \)
  - Matrices are bold uppercase: \( \mathbf{M} \)
  - Tensors are calligraphic letters: \( \mathcal{T} \)

Overview
- Vectors in \( \mathbb{R}^2 \)
- Scalar product
- Outer Product
- Bases and transformations
- Inverse Transformations
- Eigendecomposition
- Singular Value Decomposition

Warm-up: Vectors in \( \mathbb{R}^n \)
- We can think of vectors in two ways:
  - Points in a multidimensional space with respect to some coordinate system
  - Translation of a point in a multidimensional space
    ex., translation of the origin \((0,0)\)

Vectors in \( \mathbb{R}^n \)
- Notation:
  \[
  \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T
  \]
- Length of a vector:
  \[
  \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}
  \]
Dot product or scalar product

- Dot product is the product of two vectors
- Example:

\[ \mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = x_1y_1 + x_2y_2 = s \]

- It is the projection of one vector onto another

\[ \mathbf{x} \cdot \mathbf{y} = \| \mathbf{x} \| \| \mathbf{y} \| \cos \theta \]

Scalar Product

- Notation

\[ \langle \mathbf{x}, \mathbf{y} \rangle \]

\[ \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \]

- We will use the last two notations to denote the dot product

Norms in \( \mathbb{R}^n \)

- Euclidean norm (sometimes called 2-norm):

\[ \| \mathbf{x} \|_2 = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \]

- The length of a vector is defined to be its (Euclidean) norm.
- A unit vector is of length 1.
- Non-negativity properties also hold for the norm:

\[ \forall \mathbf{x} \neq 0 : \| \mathbf{x} \|^2 > 0 \quad \| \mathbf{x} \|^2 = 0 \iff \mathbf{x} = 0 \]

Bases and Transformations

- We will look at:
  - Linear Independence
  - Bases
  - Orthogonality
  - Change of basis (Linear Transformation)
  - Matrices and Matrix Operations

Linear Dependence

- Linear combination of vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \)

\[ c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \cdots + c_n \mathbf{x}_n \]

- A set of vectors \( \mathbf{X} = \{ \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \} \) are linearly dependent if there exists a vector \( \mathbf{x} \in \mathbf{X} \) that is a linear combination of the rest of the vectors.
Linear Dependence

- In $\mathbb{R}^n$
  - sets of $n+1$ vectors are always dependent
  - there can be at most $n$ linearly independent vectors

Bases

- A basis is a linearly independent set of vectors that spans the "whole space"; i.e., we can write every vector in our space as a linear combination of vectors in that set.
- Every set of $n$ linearly independent vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$
- A basis is called
  - orthogonal, if every basis vector is orthogonal to all other basis vectors
  - orthonormal, if additionally all basis vectors have length 1.

Bases (Examples in $\mathbb{R}^2$)

Bases

- Standard basis in $\mathbb{R}^n$ is made up of a set of unit vectors:
  \[
  \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
  \]
- We can write a vector in terms of its standard basis:
  \[
  \begin{pmatrix} \frac{1}{\sqrt{1 + 2}} \\ \frac{2}{\sqrt{1 + 2}} \end{pmatrix} = \hat{e}_1 + \frac{2}{\sqrt{1 + 2}} \hat{e}_2
  \]
- Observation: to find the coefficient for a particular basis vector, we project our vector onto it.
  \[
  x_i = \hat{e}_i \cdot x
  \]

Change of basis

- Suppose we have a new basis $B = \left[ b_1 \cdots b_n \right]$, $b_i \in \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$ that we would like to represent in terms of $B$
- Compute the new components
  \[
  \tilde{x} = B^{-1} x
  \]
- When $B$ is orthonormal
  \[
  \tilde{x} = b_i \cdot x
  \]
- Note the use of a dot product

Change of basis

Outer Product

- A matrix $M$ that is the outer product of two vectors is a matrix of rank 1.
Matrix Multiplication – dot product

- Matrix multiplication can be expressed using dot products

\[
BA = \begin{bmatrix}
  b_1 \\
  \vdots \\
  b_n
\end{bmatrix}
\begin{bmatrix}
  a_1 & \cdots & a_n \\
  b_1 \\
  \vdots \\
  b_n
\end{bmatrix}
= \sum_{j=1}^{n} b_j \cdot a_j
\]

Matrix Multiplication – outer product

- Matrix multiplication can be expressed using a sum of outer products

\[
BA = \begin{bmatrix}
  a_1 \\
  \vdots \\
  a_n
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  \vdots \\
  b_n
\end{bmatrix}
= b_1 a_1^T + b_2 a_2^T + \cdots + b_n a_n^T
= \sum_{j=1}^{n} b_j a_j^T
\]

Rank of a Matrix

The rank of a matrix is the number of linearly independent rows or columns.

Examples: \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) has rank 2, but \( \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \) only has rank 1.

Equivalent to the dimension of the range of the linear transformation.

A matrix with full rank is called non-singular; otherwise it is singular.

Singular Value Decomposition:

\[ D = U \Sigma V^T \]

- A matrix \( D \in \mathbb{R}^{m \times n} \) has a column space and a row space
- SVD orthogonally these spaces and decomposes \( D \)
  \[ D = U \Sigma V^T \]
  ( \( U \) contains the left singular vectors/eigenvectors
  \( V \) contains the right singular vectors/eigenvectors
  - Rewrite as a sum of a minimum number of rank-1 matrices
  \[ D = \sum_{i=1}^{\text{rank}(D)} \sigma_i u_i \circ v_i \]

Matrix SVD Properties: \( D = U \Sigma V^T \)

- Rank Decomposition:
  \[ D = \sum_{i=1}^{\text{rank}(D)} \sigma_i u_i \circ v_i \]

- Multilinear Rank Decomposition:
  \[ D = \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{ij} u_{ij} \circ v_{ij} \]

Matrix Inverse

A linear transformation can only have an inverse, if the associated matrix is non-singular.

The inverse \( A^{-1} \) of a matrix \( A \) is defined as:

\[ A^{-1}A = I \quad (= AA^{-1}) \]

We cannot cover here, how the inverse is computed.

Nevertheless, it is similar to solving ordinary linear equation systems.
Some matrix properties

Matrix multiplication $(AB)^{-1} = B^{-1}A^{-1}$.

For orthonormal matrices it holds that $A^{-1} = A^T$.

For a diagonal matrix $D = \text{diag}\{d_1, \ldots, d_n\}$,

$$D^{-1} = \text{diag}\{d_1^{-1}, \ldots, d_n^{-1}\}$$

Matlab Tutorial