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Affine-invariant geodesic geometry of deformable 3D shapes

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1. Introduction

Modeling 3D shapes as Riemannian manifold is a ubiquitous approach in many shape analysis applications. In particular, in the recent decade, shape descriptors based on geodesic distances induced by a Riemannian metric have become popular. Notable examples of such methods are the canonical forms [7] and the Gromov–Hausdorff [9,14,2] and the Gromov–Wasserstein[13,6] frameworks, used in shape comparison and correspondence problems. Such methods consider shapes as metric spaces endowed with a geodesic distance metric, and pose the problem of shape similarity as finding the minimum-distortion correspondence between the metrics. The advantage of the geodesic distances is their invariance to inelastic deformations (bendings) that preserve the Riemannian metric, which makes them especially appealing for non-rigid shape analysis. A particular setting of finding shape self-similarity can be used for intrinsic symmetry detection in non-rigid shapes [17,25,12,24].

The flexibility in the definition of the Riemannian metric allows extending the invariance of the aforementioned shape analysis algorithms by constructing a geodesic metric that is also invariant to global transformations of the embedding space. A particularly general and important class of such transformations are the affine transformations. Such transformations are a common local model for perspective distortions in images [15], and affine invariance is a necessary property of image descriptors. In 3D shape analysis, global affine transformations play an important role in paleontological research studying bones of prehistoric creatures that may be squeezed by earth pressure [8]. Furthermore, photometric properties of 3D shapes and images can be treated as embedding coordinates in high-dimensional spaces that include both geometric and color coordinates [20,11]. Photometric transformations can be thus represented as geometric transformations of the respective coordinates, for example, affine transformations in the Lab color space correspond to brightness, contrast, hue, and saturation transformations. Affine-invariant metrics are thus useful for a description of the object that is invariant to color transformations.

Many frameworks have been suggested to cope with the action of the affine group in a global manner, trying to undo the affine transformation in large parts of a shape or a picture. While the theory of affine invariance is known for many years [4] and used for curves [18] and flows [19], no numerical constructions applicable to general 2D manifolds have been proposed.

In this paper, we construct an (equi-)affine-invariant Riemannian geometry for 3D shapes. So far, such metrics have been defined for convex surfaces; we extend the construction to surfaces with non-vanishing Gaussian curvature. By defining an affine-invariant Riemannian metric, we can in turn define affine-invariant geodesics, which result in a metric space with a stronger class of invariance. This new metric allows us to develop efficient computational tools that handle non-rigid deformations and equi-affine transformations. We demonstrate the usefulness of our

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construction in a range of shape analysis applications, such as shape processing, construction of shape descriptors, correspondence, and symmetry detection.

2. Background

We model a shape \((X,g)\) as a compact complete 2D Riemannian manifold (surface) \(X\) with a metric tensor \(g\). The metric \(g\) can be identified with an inner product \(\langle \cdot, \cdot \rangle_{X}: T_{x}X \times T_{x}X \rightarrow \mathbb{R}\) on the tangent plane \(T_{x}X\) at point \(x\). We further assume that \(X\) is embedded into \(\mathbb{R}^{3}\) by means of a regular map \(x: U \subseteq \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}\), so that the metric tensor can be expressed in coordinates as

\[
\hat{g}_{ij} = \frac{\partial x^{1}}{\partial u^{i}} \frac{\partial x^{2}}{\partial u^{j}},
\]

where \(u^{i}\) are the coordinates of \(U\).

The metric tensor relates infinitesimal displacements in the parametrization domain \(U\) to displacement on the manifold

\[
dp^{2} = \hat{g}_{11} du^{12} + 2 \hat{g}_{12} du^{1} du^{2} + \hat{g}_{22} du^{2}. \tag{2}
\]

This, in turn, provides a way to measure length structures on the manifold. Given a curve \(C: [0,T] \rightarrow X\), its length can be expressed as

\[
\ell(C) = \int_{0}^{T} \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)} dt, \tag{3}
\]

where \(\dot{\gamma}\) denotes the velocity vector.

2.1. Geodesics

Minimal geodesics are the minimizers of \(\ell(C)\), giving rise to the geodesic distances

\[
d_{g}(x,x') = \min_{C \in \Gamma(x,x')} \ell(C), \tag{4}
\]

where \(\Gamma(x,x')\) is the set of all admissible paths between the points \(x\) and \(x'\) on the surface \(X\), where due to completeness assumption, the minimizer always exists.

Structures expressible solely in terms of the metric tensor \(g\) are called intrinsic. For example, the geodesic can be expressed in this way. The importance of intrinsic structures stems from the fact that they are invariant under isometric transformations (bendings) of the shape. In an isometrically bent shape, the geodesic distances are preserved—a property allowing to use such structures as invariant shape descriptors [7].

2.2. Fast marching

The geodesic distance \(d_{g}(x_{0},x)\) can be obtained as the viscosity solution to the eikonal equation \(|\nabla d_{g}| = 1\) (i.e. the largest \(d\) satisfying \(|\nabla d|_{\infty} \leq 1\)) with boundary condition at the source point \(d(x_{0}) = 0\). In the discrete setting, a family of algorithms for finding the viscosity solution of the discretized eikonal equation by means of a fast marching closely resembles the classical Dijkstra’s algorithm for the shortest path computation in graphs, with the main difference in the update step. Unlike the graph case where the shortest paths are restricted to pass through the graph edges, the continuous approximation allows paths passing anywhere in the mesh triangles. For that reason, the value of \(d(x_{0},x)\) can be obtained from the values of the distance map at two other vertices forming a triangle with \(x\). Computation of the distance map from a single source point has the complexity of \(O(N\log N)\) [23]. On parametric surfaces, the fast marching can be carried out by means of a raster scan and efficiently parallelized, which makes it especially attractive for GPU-based computation [21,3].

3. Affine-invariant geometry

An affine transformation \(x \rightarrow Ax + b\) of the 3D Euclidean space can be parametrized using 12 parameters: nine for the linear transformation \(A\), and additional three, \(b\), for a translation, which we will omit in the following discussion (here, we assume vectors to be column). Volume-preserving transformations, known as special or equi-affine are restricted by det \(A = 1\). Such transformations involve only 11 parameters. In the following, when referring to affine transformations and affine invariance, we will imply volume-preserving (equi-)affine transformations.

An equi-affine metric can be defined through the parametrization of a curve on the surface. Let \(C\) be the coordinates of a curve on the surface \(X\) parametrized by \(p\). By the chain rule

\[
\begin{align*}
C_{p} &= x_{1} \frac{du_{1}}{dp} + x_{2} \frac{du_{2}}{dp}, \\
C_{pp} &= x_{1} \frac{d^{2}u_{1}}{dp^{2}} + x_{2} \frac{d^{2}u_{2}}{dp^{2}} + x_{11} \left( \frac{du_{1}}{dp} \right)^{2} \\
&\quad + 2x_{12} \frac{du_{1}}{dp} \frac{du_{2}}{dp} + x_{22} \left( \frac{du_{2}}{dp} \right)^{2},
\end{align*}
\]

where, for brevity, we denote \(x_{i} = \partial x/\partial u_{i}\), \(x_{ij} = \partial^{2} x/\partial u_{i}\partial u_{j}\), \(C_{p} = \partial C/\partial p\), and \(C_{pp} = \partial^{2} C/\partial p^{2}\). As volumes are preserved under the equi-affine group of transformations, we define the invariant arclength through

\[
f(x) \det(x_{1}, x_{2}, C_{pp}) = 1, \tag{6}
\]

where \(f(X)\) is a normalization factor for parameterization invariance (i.e. invariance with respect to the choice of \(p\)), and the determinant is applied on a matrix formed by the column vectors \(x_{1}, x_{2}\), and \(C_{pp}\). Since \(x_{i}\) is parallel to \(x_{i}(du_{i}/dp)\) it follows that

\[
\det(x_{1}, x_{2}, Ax + b) = 0 \quad \forall a, b, \tag{7}
\]

and plugging (5) into (6) using (7) yields the equi-affine arclength

\[
dp^{2} = f(X) \det(x_{1}, x_{2}, x_{11} \frac{du_{1}}{dp} + 2x_{12} \frac{du_{1}}{dp} \frac{du_{2}}{dp} + x_{22} \frac{du_{2}}{dp})
\]

\[
= f(X) \left( \hat{g}_{11} \frac{du_{1}}{dp} + 2 \hat{g}_{12} \frac{du_{1}}{dp} \frac{du_{2}}{dp} + \hat{g}_{22} \frac{du_{2}}{dp} \right)^{2}, \tag{8}
\]

where \(\hat{g}_{11}, \hat{g}_{12}, \hat{g}_{22}\) are the determinants of the matrices \(\hat{g}_{ij}\) at \(x_{1}, x_{2}\).

In order to evaluate \(f(X)\) such that the quadratic form (8) will also be parameterization invariant, we introduce an arbitrary parameterization \(\bar{U}_{1}\) and \(\bar{U}_{2}\), for which \(x_{1} = \partial x/\partial \bar{U}_{1}\) and \(x_{2} = \partial x/\partial \bar{U}_{2}\). The relation between the two sets of parameterizations can be expressed using the chain rule

\[
\bar{U}_{1} = \underbrace{\eta_{1}}_{x_{1}}, \quad \bar{U}_{2} = \underbrace{\eta_{2}}_{x_{2}}, \tag{9}
\]

It can be shown [1,14] using the Jacobian

\[
J = \begin{pmatrix} u_{1}\eta_{1} & u_{2}\eta_{1} \\ u_{1}\eta_{2} & u_{2}\eta_{2} \end{pmatrix}, \tag{10}
\]

that

\[
\bar{U}_{1} \bar{U}_{1} = \hat{g}_{11} \frac{du_{1}}{dp} + 2 \hat{g}_{12} \frac{du_{1}}{dp} \frac{du_{2}}{dp} + \hat{g}_{22} \frac{du_{2}}{dp} \det(\bar{J}), \tag{11}
\]

and

\[
\bar{U}_{1} \bar{U}_{2} - \bar{U}_{2} \bar{U}_{1} = (\hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^{2}) \det(\bar{J}), \tag{12}
\]
where $\mathbb{g} = \det(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k)$. From (11) and (12) we conclude that
\[
\frac{\mathbb{g}_{11} \mathbf{d}u^2 + 2 \mathbb{g}_{12} \mathbf{d}u \mathbf{d}v + \mathbb{g}_{22} \mathbf{d}v^2}{\mathbb{g}_{11} \mathbb{g}_{22} - \mathbb{g}_{12}^2} = \mathbb{g}_{11} \mathbf{d}u^2 + 2 \mathbb{g}_{12} \mathbf{d}u \mathbf{d}v + \mathbb{g}_{22} \mathbf{d}v^2,
\]
(13)
and derive the affine-invariant parameter normalization
\[
f(X) = |\mathbb{g}|^{-1/4},
\]
(14)
which defines the equi-affine pre-metric tensor [4,19]
\[
\mathbb{g}_{ij} = \mathbb{g}_{ij} |\mathbb{g}|^{-1/4}.
\]
(15)
The pre-metric tensor (15) applies only for strictly convex surfaces [4]; a similar difficulty appeared in equi-affine curve evolution. There the arclength was determined by the absolute value of the geometric structure [18]. In two dimensions the problem is more acute as we can encounter non-positive definite metrics in concave and hyperbolic regions.

We propose fixing the metric by flipping the main axes of the operator, if needed. In practice, we restrict the eigenvalues of the tensor to be positive. From the eigendecomposition in matrix notation, $\mathbf{G} = \mathbf{U} / \Gamma^2$ of $\mathbb{g}$, where $\mathbf{U}$ is orthogonal and $\Gamma = \text{diag}(\gamma_1, \gamma_2)$, we compose a new metric $\mathbf{G}$, such that
\[
\mathbf{G} = \mathbf{U} / \Gamma^2 \mathbf{U}^T
\]
is positive definite and equi-affine invariant, for surfaces with non-vanishing Gaussian curvature.

4. Discretization

We model the surface $X$ as a triangular mesh, and construct three coordinate functions $x(u,v), y(u,v)$, and $z(u,v)$ for each triangle. While this can be done practically in any representation, we use the fact that a triangle and its three adjacent neighbors, can be unfolded to the plane, and produce a parameter domain. The coordinates of this planar representation are used as the parametrization with respect to which the first fundamental form is performed for every triangle of the mesh (Algorithm 1).

Algorithm 1. Equi-affine-invariant metric discretization.

**Input:** $3 \times 6$ matrix $\mathbf{P}$ of triangle vertex coordinates in $\mathbb{R}^3$ (each column $\mathbf{P}_i$ represents the coordinates of a vertex, the first three columns belonging to the central triangle).

**Output:** $6 \times 3$ matrix of coefficients $\mathbf{D}$

1. Flatten the triangles to a plane, such that each vertex $\mathbf{P}_i$ becomes $\mathbf{Q}_i \in \mathbb{R}^2$, and (i) the first vertex becomes the origin, $\mathbf{C}_1 = [0\ 0]^T$; (ii) edge lengths are preserved, $d(\mathbf{C}_i, \mathbf{C}_j) = d(\mathbf{P}_i, \mathbf{P}_j)$ for all $i$ and $j$; and (iii) the orientation is unchanged, sign $\mathbf{C}_i^T \mathbf{C}_j = \text{sign} \mathbf{P}_i^T \mathbf{P}_j$.

2. Construct a new parameterization $\mathbf{\hat{C}_i} = \mathbf{M} \mathbf{C}_i$, where

\[
\mathbf{M} = [\mathbf{C}_2 \mathbf{C}_3]^+.
\]

3. Calculate the coefficients $\mathbf{D} = \mathbf{N}^{-1} \mathbf{P}_i$ of each coordinate polynomial, where $u = \mathbf{\hat{C}}_1$, $v = \mathbf{\hat{C}}_2$, and $\mathbf{N}$ is a $6 \times 6$ matrix with each row defined as

\[
\mathbf{N}_i = [1\ u\ uv \ u^2\ v].
\]

Once the coefficients $\mathbf{D}$ are known, evaluating the equi-affine metric, as seen in Fig. 1, becomes straightforward using
\[
\mathbf{x}_i = \begin{bmatrix}
D_{31} + D_{41} v + 2 D_{51} u \\
D_{32} + D_{42} v + 2 D_{52} u \\
D_{33} + D_{43} v + 2 D_{53} u
\end{bmatrix},
\]
where $D_{ij}$ is the $(i,j)$-th entry of $\mathbf{D}$, with $D_{ij} = \mathbb{g}_{ij}^2 / \mathbb{g}^3$.

Fig. 1. The three neighboring triangles together with the central one are unfolded flat to the plane. The central triangle is canonized into a right isosceles triangle while the rest of its three neighboring triangles follows the same planar affine transformation. Finally, the six surface coordinate values at the vertices are used to interpolate a quadratic surface patch from which the metric tensor is computed.

Fig. 2. Geodesic level sets of the distance function computed from the tip of the tail, using the standard (left) and the proposed equi-affine (right) geodesic metrics.

5. Results

The equi-affine metric can be used in many existing methods that process geodesic distances. In what follows, we show several examples for embedding the new metric in known applications such as voronoi tessellation, canonical forms, non-rigid matching, and symmetry detection.

5.1. Voronoi tessellation

Voronoi tessellation is a partitioning of $(X, g)$ into disjoint open sets called Voronoi cells. A set of $k$ points $x_i \in X_{i=1}^k$ on the surface defines the Voronoi cells $V_{x_i}$, such that the $i$-th cell contains all points on $X$ closer to $x_i$ than to any other $x_j$ in the sense of the

\[
L^2(dx, dy) = g_{11} dx^2 + 2 g_{12} dx dy + g_{22} dy^2.
\]
metric $g$. Voronoi tessellations created with the equi-affine metric commute with equi-affine transformations as visualized in Fig. 4.

5.2. Canonical forms

Methods considering shapes as metric spaces with some intrinsic (e.g. geodesic) distance metric is an important class of approaches in shape analysis. Geodesic distances are particularly appealing due to their invariance to inelastic deformations that preserve the Riemannian metric.

Elad and Kimmel [7] proposed a shape recognition algorithm based on embedding the metric structure of a shape $(X,d_X)$ into a low-dimensional Euclidean spaces. Such a representation, referred to as canonical form, allows undoing the degrees of freedom due to all possible isometric non-rigid shape deformations and translating them into a much simple Euclidean isometry.
Given a shape sampled at \( N \) points and an \( N \times N \) matrix of pairwise geodesic distances, the computation of the canonical form consists of finding a configuration of \( N \) points \( z_1, \ldots, z_N \) in \( \mathbb{R}^m \) such that \( \|z_i - z_j\|_2 \approx d_X(x_i, x_j) \). This problem is known as \textit{multidimensional scaling} (MDS) and can be posed as a non-convex least-squares optimization problem of the form

\[
\{z_1, \ldots, z_N\} = \arg \min_{z_1, \ldots, z_N} \sum_{i > j} \|z_i - z_j\|_2 - \tilde{d}_X(x_i, x_j))^2.
\]  

The invariance of the canonical form to shape transformations depends on the choice of the distance metric \( d_X \). Fig. 5 shows an example of a canonical form of the human shape undergoing different bendings and affine transformations of varying strength. The canonical form was computed using the geodesic and the proposed equi-affine distance metric. One can clearly see the nearly perfect invariance of the latter. Such a strong invariance allows to compute correspondence of full shapes under a combination of inelastic bendings and affine transformations.

5.3. Non-rigid matching

Two non-rigid shapes \( X, Y \) can be considered similar if there exists an isometric correspondence \( C \subset X \times Y \) between them, such that \( \forall x \in X \) there exists \( y \in Y \) with \( (x, y) \in C \) and vice versa, and \( d_{X}(x, x') = d_{Y}(y, y') \) for all \( (x, y), (x', y') \in C \), where \( d_{X}, d_{Y} \) are geodesic distance metrics on \( X, Y \). In practice, no shapes are truly isometric, and such a correspondence rarely exists; however, one can attempt finding a correspondence minimizing the metric distortion,

\[
\text{dis}(C) = \max_{(x, x') \in C} d_{X}(x, x') - d_{Y}(y, y').
\]  

The smallest achievable value of the distortion is called the \textit{Gromov–Hausdorff distance} [5] between the metric spaces \((X, d_X)\) and \((Y, d_Y)\),

\[
d_{GH}(X, Y) = \inf_{C} \text{dis}(C),
\]  

and can be used as a criterion of shape similarity.

The choice of the distance metrics \( d_X, d_Y \) defines the invariance class of this similarity criterion. Using geodesic distances, the similarity is invariant to inelastic deformations. Here, we use geodesic distances induced by our equi-affine Riemannian metric tensor, which gives additional invariance to affine transformations of the shape.

Bronstein et al. [2] showed how (19) can be efficiently approximated using a convex optimization algorithm in the spirit of multidimensional scaling (MDS), referred to as generalized MDS (GMDS). Since the input of this numeric framework is geodesic distances between mesh points, all that is needed to obtain an equi-affine GMDS is one additional step where we substitute the geodesic distances with their equi-affine equivalents. Fig. 6 shows the correspondences obtained between an equi-affine transformation of a shape using the standard and the equi-affine-invariant versions of the geodesic metric.

5.4. Intrinsic symmetry

Raviv et al. [17] introduced the notion of \textit{intrinsic symmetries} for non-rigid shapes as self-isometries of a shape with respect to a deformation-invariant (e.g., geodesic) distance metric. These self-isometries can be detected by trying to identify local minimizers of the metric distortion or other methods proposed in follow-up publications [16,25,12,24].

Here, we adopt the framework of [17] for equi-affine intrinsic symmetry detection. Such symmetries play an important role in paleontological applications [8]. Equi-affine intrinsic symmetries are detected as local minima of the distortion, where the equi-affine geodesic distance metric is used. Fig. 7 shows that using the traditional metric we face a decrease in accuracy of symmetry detection as the affine transformation becomes stronger (the accuracy is defined as the average geodesic distance between the detected and the ground-truth symmetry). Such a decrease does not occur using the equi-affine metric.

6. Conclusions

We introduced a framework for the construction of (equi-) affine-invariant Riemannian metric and the associated geodesic geometric, and showed that it can be utilized to construct affine-invariant shape
It is important to note that our construction addresses affine invariance locally though the construction of a Riemannian metric, which in theory would allow invariance to a more generic class of spatially varying affine transformations. Such a situation is typical in image analysis, where affine transformations are a local model for more general view point transformations.

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**References**