1) Describe a few of the most important developments in the history of dynamical systems theory.

Understanding dynamical systems has required the development of many tools that radically altered our corresponding understanding of the physical world and mathematics itself. The tools of infinitesimal calculus, the principle of least action, and topological descriptions of the qualitative behavior of dynamical systems are three of the most important contributions that dynamical systems theory has provided. These have greatly increased our insight into the causes and forces that give rise to motion.

*Infinitesimal Calculus*

Probably the most widely applied notion developed by those trying to understand mechanics is that of infinitesimal calculus. While many problems in statics could be understood without the aid of calculus, the dynamic problems of celestial mechanics and the computation of accurate *ephemerides* demanded more complex tools.

The roots of infinitesimal calculus can be traced to the work of the ancient Greek geometers. The notion of *quadrature* of a shape, that is finding a square whose area equaled the shape, was of interest to Hellenic mathematics. Struik remarks that the "problem of the quadrature of the circle is one of the 'three famous problems of antiquity'" which captivated the Greeks. The reason that these problems were so difficult is that they required the geometers to confront issues that "cannot be geometrically solved by the construction of a finite number of straight lines and circles except by approximating" [Struik, 1967]. In short, they forced Greek mathematics to confront notions of the infinite that initially led to conceptual difficulties embodied in Zeno's paradoxes.

The "exhaustion method" (whose creation is credited to Eudoxus) helped the Greeks avoid Zeno's paradoxes but did not surmount the problems raised by Zeno altogether. Struik comments "It avoided the pitfalls of the infinitesimals by simply discarding them, reducing problems which might lead to infinitesimals to problems involving formal logic only." Archimedes deftly employed Eudoxus' technique; his *Quadrature of the Parabola* provided demonstrations "based on the method of exhaustion" [Ball, 1960]. Archimedes found the area and volume of spheres and "quadratic surfaces of revolution" by making "consistent use of the exhaustion method to prove the results of his integration" [Struik, 1967].

Consideration of quadrature would become very important to later mathematicians who were interested in developing symbolic methods for quadrature or integration. The Greeks were also interested in the *tangency problem*. This was important because tangency is related to differentiation in the same manner that quadrature is related to integration. Apollonius was notable for introducing and solving the problem of "the construction of circles tangent to three given circles" using "compass and ruler only" [Struik, 1967]. Ultimately the geometrical notions of quadrature and tangency would provide a basis for the development of the symbolic formulation of integration and differentiation.

As the practice of physics became firmly rooted in empiricism, natural philosophers found that dictates of Aristotelian physics did not match observation. They found instead that the motion of falling objects was uniformly accelerated. Truesdell forcefully argues that "the main kinematical properties of uniformly accelerated motion, still attributed to Galileo by the physics texts, were discovered and proved by the scholars of Merton College -- William Heytesbury, Richard Swineshead, and John Dumbleton -- between 1328 and 1350." Furthermore "their work distinguished *kinematics*, the geometry of motion, from *dynamics*, the theory of the causes of
motion." Their work set the tone for much of the later development of dynamics but also "foreshadowed the concepts of function and derivative" [Truesdell, 1968].

Galileo picked up the problem of uniformly accelerated motion and furthermore gave an "argument to show that the area under the time-velocity curve is distance" thus relating quadrature and integration of velocity. Cavalieri, "a pupil of Galileo," pushed the notion further by exploring geometrical methods to treat indivisibles. He held that "a line is made up of points as a string is of beads" and that shapes could be composed of "an infinite number of constituent elements" [Kline, 1972]. In doing so he approached what the Greeks had deftly avoided. He also brought the tangency problem to the forefront of mechanics. His work was extended by Wallis' *Arithmetica infintorum*, which provided "the most advanced form of calculus in the period before Newton and Leibniz" [Struik, 1967]. Huygens also contributed to the development of calculus with the study of pendulums. However neither Huygens nor Wallis were able to completely familiarize themselves with the more formal developments of Newton and Leibniz. Barrow (of whom Newton was a pupil) developed geometrical methods to tackle the tangency problem but also saw tangency (differentiation) and quadrature (integration) as inverses of one another.

Newton and Leibniz share the credit for the actual "invention" of calculus but their formulations were quite distinct from one another, though equivalent. Newton conceived fluxions as geometrical tangents and fluents as a way for computing the quadrature. But he also saw clearly how these related to differential equations. Using this machinery he was able to refashion Kepler's laws into an elegant Laws of Motion:

"I. Every body continues in its state of rest, or of uniform motion straight ahead, unless it be compelled to change that state by forces impressed upon it

II. The change of motion is proportional to the motive force impressed, and it takes place along the right line in which that force is impressed.

III. To an action there is always a contrary and equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts" [Truesdell, 1968].

The importance of this fluxional formulation is not to be understated; "from a few simple axioms, all the major properties of the motions of bodies had been proved" [Truesdell, 1968]. Or, as Ball more pleasingly says: "No sooner had Newton proved this superb theorem ... than all the mechanism of the universe at once lay spread before him."

Leibniz's formulation was much more symbolic, and proved to be quite useful to those who were to start making use of infinitesimal calculus. Leibniz published his calculus in 1684 manuscript entitled *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*. His notation (dx, dy and the integration symbol) is still used, as are his terms: calculus, differentiation, integration, function, and coordinates. In 1687 the Bernoulli brothers began to make use of his methods. Johann Bernoulli further extended calculus with the invention of the calculus of variations. The Bernoullis along with Euler are credited with formalizing and developing Leibniz and Newton's infinitesimal calculus as well as applying it to numerous problems in dynamics [Struik, 1968].

*The Principle of Least Action*

The principle of least action is one of the most elegant notions developed to describe how dynamical systems evolve. The first descriptions of the principle actually predate the calculus of variations, which provides a formal argument for nature’s adherence to it. Leanoardo da Vinci’s notebooks foreshadow the concept when he observes: "every action done by natures is done in the shortest way" [Truesdell, 1968]. Truesdell pessimistically suggests this be taken lightly
because Leonardo is prone to casually stating conjectures without further experimentation. But Truesdell does admit that "enthusiasts have interpreted this pronouncement as implying Fermat's principle of least time in optics and Maupertuis' principle of least action in analytical dynamics" [Truesdell, 1968]. The actual term "action" was used first by Huygens to refer to "the effect of a motion." Leibniz elaborated on the concept in *Dynamica* when he spoke of least action with respect to "harmless action" [Sussman and Wisdom, 2001].

As we see in the quote above, Maupertuis is most frequently credited with the development of the principle of least action. In 1750 he "entered into a spirited controversy with the Swiss mathematician Samuel Koenig" [Struik, 1968]. Maupertuis' formulation "defined as his 'action' the quantity mv (m = mass, v = velocity, s = distance)" [Struik, 1968]. However he was ridiculed for combining this with "a proof for the existence of God." However, Euler saw the value of the formulation and "defended and elaborated the theory of 'least action' which had been propounded by Maupertuis in 1751 in his *Essai de cosmologie*" [Ball, 1960].

Lagrange's formulation of a path-distinguishing function explicitly built off this notion. He applied the principle of least action to describe the path that would be realized by a dynamical system from a configuration. Action, as used by Lagrange, was more specifically defined to be the integral of the *vis viva* (or twice the kinetic energy) [Sussman and Wisdom, 2001]. Hamilton's reformulation of the Lagrangian in terms of momenta and coordinates continued use of the principle of least action. Others have since recast the notion of as the principle of stationary action to indicate that the action remains stationary along the variational path contours that bracket the actual path.

**Topological Dynamics**

Lagrange's deft use of analytical methods crystallized the practice of dynamics. His methods were restricted to algebraic analysis, as the preface to his *Mécanique Analytique* noted:

> The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure [Truesdell, 1968].

However, in working on the three-body problem, Poincaré discovered limitations to this approach. He found that it would be impossible to find "new and independent integrals that would enable further reduction" of the problem [Gorroff, 1993]. This led Poincaré to consider what sort of qualities one could prove about dynamical systems. He returned to the geometrical formulations first used in the development of infinitesimal calculus to investigate these properties and began to develop a more topological approach to analysis.

His use of first return maps and phase portraits set an entirely new agenda for the study of dynamical systems. Mathematicians who followed his approach sought to understand the global behavior of dynamical systems reconceived as manifolds. Liapounov contributed notions of the stability of topological representations of dynamical systems. Birkhoff contributed to the understanding of chaotic behavior that Poincaré first observed [Abraham and Shaw, 1982]. Peixoto provided early results on the global behavior of manifolds that were further formalized by Smale. Moser's work on twist maps along with Kolmogorov, Arnol'd, and Moser's KAM theorem showed the conditions under which chaos was limited [Moser, 1973]. More recently Thom has investigated catastrophic behavior of manifolds that arises when attractors disappear altogether [Ekland, 1988].

Each of the developments detailed above had a substantial impact on the progression of dynamical systems theory. The developments were not invented and perfected by any one mathematician but structured and organized the efforts of many formidable thinkers. They constitute a common thread visible as we look back upon the difficult problems posed by efforts to understand the causes of motion.
2) What is a dynamical system? What is an adaptive system? How do the concepts and tools for one help understand the other?

A dynamical system is a system that provides a map from a state space onto itself; describing its evolution. This means that if we have some function \( f(t, x) \) which describes the state of the system at time \( t \) starting at coordinates \( x \), then there is some function \( f(s, f(t, x)) = f(s+t, x) \) that describes the state of the system at time \( s+t \). If our state space is composed of real numbers then our system is a continuous and can be described in terms of differential equations that give the state at the next instant. If our state space is instead composed of integers, then our system is instead a discrete dynamical system that described in terms of a difference equation. Such a system is more specifically called a differentiable dynamical system: if the state space is considered topologically as a manifold, then the map must be a diffeomorphism [Smale, 1986].

If the state is known with perfect accuracy then dynamical systems are deterministic implying that the complete state at any future time can be computed. However, it is important to be mindful that small differences in initial conditions can lead to wildly different configurations at future times.

In contrast, an adaptive process is something that gradually makes an object (organism) more suitable to some other object (environment). One could say it iteratively creates adaptors that (over time) transition the configuration of a system from one state towards a more suitable state. This requires that we have a formal notion of "suitable" often encoded in an objective function. There must be some function of the organism object and the environment object that tells us how suitable the organism is within the environment.

Some prominent examples of adaptive behavior are genetic algorithms, simulated annealing, and negative feedback within electrical circuits. Each has a different way of encoding what is suitable. In a genetic algorithm, suitability is encoded in a fitness function used to evaluate organisms. In simulated annealing the suitability is thought of as a "ground state" of a thermodynamic system [Kirkpatrick et al., 1983]. In the case of negative feedback the notion of what is suitable is instead encoded as an attractor toward which the systems tends.

Some of the above are certainly differentiable dynamical systems while others require a different formulation altogether. Negative feedback in electrodynamical systems can be deterministically described and has all of the properties of a differentiable dynamical system. The state at some future time can be computed by solving the initial value problem. However a genetic algorithm does not provide a completely deterministic model of genetic behavior. Point mutations and crossovers are considered to be random processes. As an aside, it is interesting to consider that if a biological process is considered as an enormous physical system and modeled as an N-body problem a deterministic description could arise. However traditional genetic algorithms that exhibit adaptive behavior are stochastic. Similarly simulated annealing makes use of statistical mechanics and Montecarlo processes to arrive at suitable thermal equilibria, making it nondeterministic. So adaptive behavior is not limited to deterministic behavior but instead includes nondeterministic systems for which "the past history determines various possibilities for the present state, among which the system will choose in response to external stimuli, such as changes in the environment" [Ekland, 1988].

This means that the machinery of differential equations cannot be uniformly applied to understand adaptive systems. Because the systems can incorporate random processes or have functions on state spaces that are not analytical, the first order differential equations used to describe the state of the system in the immediate future do not exist in every case. To use the map analogy, one can have a map which points from a point in configuration space not to another single point but instead to a whole set of possible points that are selected between using extrinsic information.

However, some tools applied to understand dynamical systems can give us interesting insights into the behavior of adaptive systems. The qualitative tools developed for global analysis of dynamical systems can help us understand the properties of adaptive systems. We can take a
genetic algorithm for instance and consider its configuration space in much the same way that
Poincaré considered the problems of celestial mechanics. Instead of thinking of the first-return
map of an orbit, we can think about the subspace that is reached by all possible mutations and
recombinations of two genetic configurations. We can imagine a two dimensional state space of
all possible genetic configurations encoded by a chromosome. Consider two organisms who are
suitable according to our fitness function. They are two points upon the two-dimensional
configuration plane. When we consider all of the points that are possible to reach by recombining
the genetic information of the organisms we sketch out a rectangle that is bounded by the corners
in which the parents reside. We cannot say exactly where the map will point in configuration
space, but we know that the child organism will reside somewhere in this rectangle. Inclusion of
mutation somewhat complicates matters, but we can still draw a picture of all the possible future
configurations, and even describe with what probability they can be reached.

Simulated annealing can similarly be described in a more qualitative light. The cooling schedule
employed to gradually anneal the system in order to reach non-degenerate and highly suitable
ground states constrains the subspace of configuration space that can be reached. When the
simulated annealing starts, large contributions toward the ground state are accepted. As the
cooling progresses, gradually smaller contributions are accepted. In our geometrically analogy,
we randomly start at some point in the configuration space. At first, our warm system in
metaphorically "melted" allowing for our point to travel large distances (we could think of this as a
area inscribing the entire configuration space). As the system cools, the area of possible states
that can be considered shrinks. As we get closer to the true ground state of the system, we
consider only possibilities in the immediate neighborhood of the most suitable configuration.

Thinking in this qualitative, geometrical manner can allow us to construct different adaptive plans
to search for suitable outcomes. Depending on the properties of our configuration space different
approaches can be employed to construct new mappings from points to subspaces. These maps
and the suitability function essentially describe the adaptive system's behavior. While this
description does not have the advantage of being deterministic, it does allow for characterization
and greater insight into adaptive processes. Such insight eventually leads to a broader
understanding of the commonalities between adaptive and dynamic phenomena.

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