DATAREDUCTION IN SPATIALLY COLORED NOISE USING A VIRTUAL UNIFORM LINEAR ARRAY

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ABSTRACT

This paper focuses on beamspace DOA estimation in colored noise. We present a beamspace transformation that preserves the Cramér Rao Bounds for the parameters of interest, and at the same time interpolates the true array manifold in such a way that a virtual uniform linear array is obtained in beamspace. Besides the computational savings offered by the dimensionality reduction, the virtual array enables the use of computationally simple algorithms such as root-MUSIC or ESPRIT.

1. INTRODUCTION

By applying beamspace transformations in array processing, it is possible to reduce the computational complexity of the applied signal processing algorithms and at the same time obtain other benefits such as reduced sensitivity to directional interference, reduced bias etc. A beamspace transformation is a linear transformation of the data from the complex m-dimensional vector space, \( \mathbb{C}^m \), (elementspace, \( m \) being the number of sensor elements) to the complex n-dimensional vector space \( \mathbb{C}^n \) (beamspace), where \( n \leq m \). A number of design techniques have appeared in the literature, see e.g. [1, 2, 3, 4]. This paper focuses on the colored noise case and presents a beamspace transformation that preserves the Cramér Rao Bounds, CRB, for the parameters of interest. Furthermore, the beamspace transformation is designed in such a way that a virtual uniform array is obtained in beamspace. This facilitates a further reduction in computational complexity, since one can employ, e.g., the root-MUSIC or the ESPRIT approach when estimating the directions of arrival, DOA:s.

2. PROBLEM FORMULATION AND THEORETICAL RESULTS

We begin this section by describing the employed signal model. An array of \( m \) sensors receives \( p \) planar, narrowband, waveforms from the directions \( \{ \theta_1, \ldots, \theta_p \} \). The complex envelope of the sensor outputs are modeled by the relation

\[
\tilde{y}(t) = \tilde{A}(\theta) s(t) + \tilde{e}(t),
\]

where \( \tilde{A}(\theta) \) is the array steering matrix, and the source signals are collected in the vector \( s(t) \). The signal \( s(t) \) and the noise \( \tilde{e}(t) \) are independent, temporally white, zero-mean, complex Gaussian random variables with second order moments

\[
E[s(t)s^*(t)] = R_{ss},
\]

\[
E[\tilde{e}(t)\tilde{e}^*(t)] = \tilde{Q}.
\]

The matrix \( \tilde{Q} \) is Hermitian and positive definite, but otherwise arbitrary. The rank of the signal covariance matrix, \( R_{ss} \), is \( d \leq p \) which covers the case of spatially correlated signals, due to, e.g., multipath etc. The notation \( \tilde{x} \) is used to designate that the quantity, \( x \), belongs to elementspace. The array covariance matrix is given by

\[
\tilde{R} = \tilde{A}(\theta) R_{ss} \tilde{A}^*(\theta) + \tilde{Q}.
\]

To avoid ambiguous parameter estimates, a unique parameterization of the array covariance matrix must be made, see, e.g., [5]. However, regardless of the actual parameterization, unambiguous or not, we can still derive a transformation that preserves the CRB. Therefore, we do not treat the parameter identifiability problem herein. Denoting the \( m \times n \) beamspace transformation by \( T \), the beamspace signal model becomes

\[
y(t) = T^* \tilde{A}(\theta) s(t) + T^* \tilde{e}(t)
\]

\[
= A(\theta) s(t) + e(t).
\]
Based on $N$ snapshots of the beamspace data, $y(t)$, we want to estimate the DOAs, $\theta = [\theta_1, \ldots, \theta_p]^T$ with as good accuracy as possible. To facilitate this, the following theorem states a condition on the range space of the beamspace transformation that is sufficient to guarantee that the beamspace CRB equals the elementspace CRB.

Theorem 1 The CRB for $\theta$ in beamspace is equal to the CRB in elementspace, provided that the beamspace transformation matrix $T$ satisfies the following condition

$$ \mathcal{R}(T) \supseteq \mathcal{R}(Q^{-1}U(\theta)) = \mathcal{R} \left( \hat{R}^{-1}U(\theta) \right), $$

where $U$ is defined as

$$ U(\theta) \triangleq \begin{bmatrix} \bar{a}(\theta_1) & \ldots & \bar{a}(\theta_p) & \bar{d}(\theta_1) & \ldots & \bar{d}(\theta_p) \end{bmatrix}, $$

and $\mathcal{R}(X)$ denotes the range space of $X$. In the above, $\bar{a}(\theta)$ is the steering vector of the array while $\bar{d}(\theta)$ is the derivative of the steering vector with respect to $\theta$.

Proof: See [6].

In beamspace DOA estimation, one usually designs the transformation to focus on a particular sector, or sectors, of interest. In the simulations of Section 5, we focus on the common problem of how to reduce the influence of "out-of-band" sources, i.e., sources lying outside the interesting sectors. We may think of these sources as interfering sources, which color the noise. In effect, we have a spatially colored noise problem.

3. A VIRTUAL UNIFORM LINEAR ARRAY TRANSFORMATION

It is the purpose of this section to derive a transformation matrix that has the property given in Theorem 1, but at the same time allows the use of a virtual linear array model, see [7]. To proceed, we assume that we are given an orthogonal basis, $V$ (with dimension $m \times n_v$), for the range space of the matrix $\hat{R}^{-1}U(\theta)$. Furthermore, we define a set of $q$ fictitious directions of arrival covering the interesting sector(s), $\theta_f = \{\theta_{f1}, \ldots, \theta_{fq}\}$, for which we calculate the original array manifold, $A(\theta_f)$, and the virtual ULA manifold, $A_v(\theta_f)$, see also Section 4. We now propose the following criterion for finding the desired beamspace transformation, $T$.

$$ T_{opt} = \arg \min_{T} \left[ (1 - \lambda) \left\| \hat{A}^*(\theta_f)T - A_v(\theta_f) \right\|_{W_1}^2 + \min_{M} \lambda \left\| T - VM \right\|_F^2 \right], 0 < \lambda < 1 $$

(6)

where $M$ is an $n_v \times n$ matrix expressing the desired linear dependence of $T$ on $V$. The matrix $W_1, q \times q$, is a positive definite, (diagonal) weighting matrix to be defined subsequently. With the formulation in (6), we have intentionally relaxed the requirement that the range of $T$ should be exactly equal to that of $V$. This can be justified by the observation that we can only estimate an $n_v \times n$ matrix, which implies that we may allow the range space of $T$ to vary slightly around the range of $V$, and at the same time maintain a better fit to the array response. However, to assure that the beamspace parameter estimates have a variance close to zero, $\lambda$ should be close to one.

The minimization problem defined in (6) can be recast in the following form (suppressing the dependence on $\theta_f$ for notational convenience)

$$ \min_{T,M} \left\| \begin{bmatrix} \sqrt{1 - \lambda} W_1^{1/2} \hat{A}^* \left[ \begin{array}{c} \sqrt{1 - \lambda} W_1^{1/2} A_v^* \end{array} \right]^T \end{bmatrix} \right\|_{VM}^2 $$

(7)

For a given $T$, (7) can be solved with respect to $M$. The solution is readily obtained as

$$ M = (V^*V)^{-1} V^*T = V^*V, $$

(8)

where the last equality follows since $V$ is a unitary matrix. Inserting the result into (7) and re-arranging terms gives

$$ \min_{T} \left\| FT - G \right\|_F^2, $$

(9)

where we have made the following definitions:

$$ F = \begin{bmatrix} \sqrt{1 - \lambda} W_1^{1/2} \hat{A}^* \end{bmatrix} $$

(10)

$$ G = \begin{bmatrix} \sqrt{1 - \lambda} W_1^{1/2} A_v^* \ 0 \end{bmatrix} $$

(11)

In Equation (10), $P_V^T = I - P_V$, where $P_V = VV^*$ is the projection onto the column space of $V$, i.e., the range space of $\hat{R}^{-1}U(\theta)$. The optimal transformation matrix is now given by

$$ T_{opt} = F^+G, $$

(12)

$X^+$ denoting the pseudo inverse of $X$.

4. IMPLEMENTATION ISSUES

We start this section by giving an algorithm for obtaining an orthogonal basis $V$ whose span approximates the span of $\hat{R}^{-1}U(\theta)$.

1. Determine a set of interesting angle intervals that will be processed in beamspace. Define also the following quantity

$$ U_f^{\theta} \triangleq \begin{bmatrix} \bar{a}(\theta_{f1}) & \ldots & \bar{a}(\theta_{fq}) \end{bmatrix}, $$

(13)
where \(\hat{a}(\theta_{f_i}), i = 1 \ldots q\), is a dense set of fictitious array steering vectors located within the selected angle intervals.

2. Calculate the quantity

\[ \hat{R}^{-\frac{1}{2}} U_f W_2, \]  

(14)

where \(W_2\) is a diagonal weighting matrix to be defined subsequently.

3. Decide on a beamspace dimension, denoted \(n_s\), that is large enough to accommodate for the maximum number of expected sources within the sectors.

4. Apply the singular value decomposition, SVD, to the quantity in (14). Calculate

\[ V = \hat{R}^{-\frac{1}{2}} [ q_1 \ldots q_{n_s} ], \]  

(15)

where \(n_s\) is the number determined in Step 3 and \(q_i, i = 1 \ldots n_s\), are the corresponding left singular vectors.

5. Orthogonalize \(V\).

Step 1 can preferably be based on the Capon spectrum, since this is used in the design of the beamspace transformation, see below. The choice of \(n_s\) in step 3 may, e.g., be based on prior information on the maximum number of sources present in the sectors, or such an estimate may be obtained with, e.g., the minimum description length principle, MDL. principle, [8]. The introduction of the weighting matrix, \(W_2\), in Step 2 is pertinent to the success of choosing the proper subspace dimension that the beamspace transformation shall preserve. With \(W_2\) equal to the identity matrix, a column of the matrix defined in (14) has the magnitude

\[ \| \hat{R}^{-\frac{1}{2}} \hat{a}(\theta_{f_i}) \|^2 = \hat{a}^*(\theta_{f_i}) \hat{R}^{-1} \hat{a}^*(\theta_{f_i}), \]

which is equal to the inverse of the Capon spectrum.

Thus, with no weighting, there is a risk that the signal part will be lost in the SVD truncation process, see Step 4. Instead, we propose using the Capon weighting. With this weighting, which is obtained by creating a weighting matrix with the Capon spectrum on the diagonal, a column of the matrix in (14) has the magnitude

\[ \| \hat{R}^{-\frac{1}{2}} \hat{a} - \frac{1}{\hat{a}^* \hat{R} \hat{a}^*} \| \hat{a}^* \hat{R} \hat{a}^* \|^2 = \frac{1}{\hat{a}^* \hat{R} \hat{a}^*} \]  

(16)

which equals the Capon spectrum. With this weighting, the above mentioned risk is greatly reduced.

Finally, the virtual ULA transformation matrix is computed according to (12). Here, we use the same fictitious DOA:s as defined in step 1 above, i.e., set \(\hat{A}(\theta_f) = U(\theta_f)\) and compute the corresponding beamspace ULA steering matrix, \(A_s(\theta_f)\). Also in this step, we define a “Capon like” weighting matrix, \(W_1\), to emphasize the matching of the virtual array response in the regions where we expect the sources to be located. However, we introduce a real constant, \(k\), that influences the sharpness of the weighting according to

\[ W_1 = W_2^k, \]  

(17)

where \(W_2\) is the Capon weighting matrix. The influence of the constant \(k\) will be seen in Section 5.

5. SIMULATIONS

In our simulation examples, we employ a uniform linear array with 25 sensors, separated by half a wavelength. Two source signals are located at \(-2^\circ\) and \(0^\circ\) within the interesting sector, defined by \([-5^\circ : 5^\circ]\) relative broadside of the array. There are three sources located outside the sector, the first at \(13^\circ\), the second at \(18^\circ\) and the location of the third is swept from \(-19^\circ\) to \(-5^\circ\) in steps of \(2^\circ\). The sources outside the sector are referred to as “out-of-band” sources. Note that the out-of-band sources act as interference. The SNR of the sources are equal to and is set to \(10\) dB relative the noise power in one element. The number of snapshots is 200 and the number of Monte Carlo trials is 100. Two different beamspace dimensions, 4 and 6, are used in the examples. The number of sources in beamspace is estimated by the MDL principle. After obtaining an estimate of the number of sources, the DOA:s are estimated using a white noise model and the root-MUSIC approach. The value of \(\lambda\) is 0.9999.

In Figure 1, the root mean square error for the source located at \(-2^\circ\) is plotted as a function of the location of the moving out-of-band source for the proposed method and the method presented in [7]. The beamspace dimension is here 4. The proposed method clearly outperforms the method of [7]. Two different weightings, \(k = 1\) and \(k = 2\), are used as discussed in 3. We see that the accuracy of the estimate depends on the value of \(k\). A higher value of \(k\) reduces the bias of the estimates, since a better fit is obtained between the true array manifold and the virtual one at the peaks of the Capon spectrum. In Figure 2, the beamspace dimension is increased to 6. The increase in the number of virtual ULA elements implies a better fit to the
true array manifold, which reduces the need for heavy weighting. A good result is obtained with $k = 1$. The RMS of the estimates obtained with the method of [7] is also lower here. A simulation for the method of [7] using the same weighting as in the presented method, i.e. $k = 1$, was also performed to obtain a better fit to the true array manifold at the expected source locations. However, no improvement over the result in Figure 2 was observed. In general, we see that the RMS of the estimates of the proposed method are very close to the square root of the CRB, provided that a suitable weighting is used.

6. CONCLUSIONS

We have presented a practical design procedure for finding a beamspace transformation that approximately preserves the Cramér Rao Bounds for the parameters of interest, and at the same time interpolates the true array manifold in such a way that a virtual uniform linear array is obtained in beamspace. This enables the use of the root-MUSIC or ESPRIT approach, which offers additional reduction in computational complexity besides the reduction obtained by the transformation of the data to a lower dimension. The simulation show that parameter estimates with a variance very close to the bound is obtained if the outlined approach is followed.

7. REFERENCES