The Complex Cepstrum of Higher Order Cumulants and Nonminimum Phase System Identification

RENLONG PAN, STUDENT MEMBER, IEEE, AND CHRYSOSTOMOS L. NIKIAS, SENIOR MEMBER, IEEE

Abstract—A new computationally efficient identification procedure is proposed for a non-Gaussian white noise driven linear, time-invariant, nonminimum phase system. The method is based on the idea of computing the complex cepstrum of higher order cumulants of the system output. In particular, the differential cepstrum parameters of the nonminimum phase impulse response are estimated directly from higher order cumulants via least-squares solution or two-dimensional FFT operations. The method reconstructs the minimum and maximum phase impulse response components separately. It is flexible enough to be applied on AR, MA, or ARMA systems without a priori knowledge of the type of the system. Benchmark simulation examples demonstrate the effectiveness of the method even with "short" length data records.

I. INTRODUCTION

NONMINIMUM phase system (or signal) identification is an important problem which arises in many signal processing applications. The available data are assumed to be the result of a linear convolution between the nonminimum phase impulse response of the system and a white noise input sequence which is unknown. The objective is to reconstruct either the impulse response or the frequency transfer function (magnitude and phase) from the available data. For example, this problem arises in data communications when a receiver is obliged to achieve a blind starting phase, i.e., without the transmission by the transmitter of a known sequence of data [1], [2]. It also arises in seismic and other deconvolution problems (see [3] and references therein) [4], [5]. The classical system identification procedures, which are almost exclusively based on least-squares estimation criteria, exploit the autocorrelation function of the available output data and thus are incapable of identifying correctly the nonminimum phase structure of the system [6], [7]. An accurate system (or signal) reconstruction in the autocorrelation (or power spectrum) domain can only be achieved if the system is indeed minimum phase.

One of the early approaches for nonminimum phase signal reconstruction has been homomorphic filtering based on complex cepstrum [3], [5], [8] or differential cepstrum [9], [10] operations. However, the cepstrum approaches require that the input signal (or reflectivity series) is an impulse train (not a stationary random process) and that it completely separates from the impulse response of the system (wavelet) in the cepstrum domain. They may also be very sensitive to observation noise [3].

For moving average (MA) processes of order one (special case), it has been demonstrated that the use of absolute value norm (L1) provides an estimate of the impulse response shape with correct phase character when the driving process is non-Gaussian [11]. Assuming knowledge of the non-Gaussian distribution, a general robust identification procedure has been proposed in [2]. However, using this procedure, the nonminimum phase signal reconstruction can only be achieved by solving a highly nonlinear system of equations.

Higher order spectra, which are defined in terms of higher order cumulants, have been given a lot of attention lately due to their ability to preserve the true phase character of non-Gaussian parametric signals. Assuming that the input white noise sequence is zero-mean non-Gaussian, the nonminimum phase impulse response (or system) can be reconstructed from the output data in a higher order spectrum domain (bispectrum or trispectrum) [12]. Several methodologies have been developed based on conventional higher order spectrum estimation techniques [13]–[15] or based on autoregressive (AR) [16]–[18], moving average (MA) [19]–[23], and ARMA models [24]–[30].

We address in this paper the reconstruction problem of a nonminimum phase impulse response sequence based on the idea of applying cepstrum operations on the higher order cumulants of the output sequence. By doing so, the method reconstructs the minimum and maximum phase components of the impulse response separately, in a computationally efficient manner. The method is flexible enough to accommodate a general ARMA non-Gaussian process and it does not require model order selection criteria.

The organization of the paper is as follows. The problem definition is formulated in Section II. The complex cepstrum of third moments and its properties are given in Section III. Section IV discusses the key cepstral equation and its usefulness for nonminimum phase signal reconstruction problems. The reconstruction methods are described in Section V, and the resulting algorithms in Section VI. Results of computer simulations are presented in
Section VII. Concluding remarks are drawn in Section VIII. The extension of these results to the case of fourth-order cumulants (trispectrum) is described in Appendix C.

We assume that the reader is familiar with the general definition of higher order spectra and their properties. They can be found in the tutorial paper [12].

II. Problem Definition—Preliminaries

Assuming that the output sequence \( \{X(k)\} \) is generated by an ARMA system described by

\[
X(k) = - \sum_{i=-l_1}^{l_1} d_i X(k - i) + \sum_{i=-l_2}^{l_2} u_i W(k + i) \tag{1}
\]

where \( \{W(k)\} \) is zero-mean non-Gaussian, white, i.i.d., with \( E\{W(k)\} = 0 \) and \( E\{W(k) W(k + \tau)\} = \delta(\tau) \) and \( E\{W(k) W(k + \tau) W(k + \rho)\} = \beta \delta(\tau, \rho) \), the problem is to reconstruct the magnitude and phase response of the system from the given output data \( \{X(k)\} \). Let us note that \( \delta(\tau) \) and \( \delta(\tau, \rho) \) are the 1-D and 2-D Kronecker delta functions, respectively.

The ARMA transfer function is given by

\[
H(z) = |H(\omega)| \exp j\Phi(\omega)
\]

or for \( z = \exp j\omega \)

\[
H(\omega) = |H(\omega)| \exp j\Phi(\omega) \tag{2.1}
\]

where \( |H(\omega)| \) and \( \Phi(\omega) \) are the magnitude and phase responses, respectively, or equivalently,

\[
H(\omega) = \frac{\sum_{i=-l_1}^{l_1} u_i \exp j(\omega i)}{1 + \sum_{i=-l_2}^{l_2} d_i \exp j(-\omega i)} \tag{2.2}
\]

Since the system is generally nonminimum phase and stable, its transfer function may also be written in terms of poles and zeros as follows:

\[
H(z) = I(z^{-1}) \cdot 0(z) \cdot A \cdot z^{-r} \tag{3.1}
\]

where \( A \) is a constant, \( r \) is integer,

\[
I(z^{-1}) = \prod_{i=1}^{l_1} \left( 1 - a_i z^{-1} \right) = \prod_{i=1}^{l_2} \left( 1 - c_i z^{-1} \right) \tag{3.2}
\]

is the minimum phase component with poles \( \{c_i\} \), and zeros \( \{a_i\} \) inside the unit circle, i.e., \( |c_i| < 1 \) for all \( \{i\} \), and

\[
0(z) = \prod_{i=1}^{l_2} (1 - b_i z) \tag{3.3}
\]

is the maximum phase component with zeros outside the unit circle, i.e., \( |b_i| > 1 \) for all \( \{i\} \). The zeros on the unit circle will cancel out in the third moment (for fourth-order cumulant) sequence. Thus, neither \( A \) nor \( r \) in (3.1) will be recoverable. Let us also note that there is no specific reason for excluding maximum phase poles in (3.3).

The method described in the paper would work for maximum phase poles too. Similarly, the minimum phase impulse response of the ARMA system is given by

\[
i(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} I(-\omega) \exp j(\omega n) d\omega \tag{4.1}
\]

and the maximum phase component

\[
o(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} 0(\omega) \exp j(\omega n) d\omega \tag{4.2}
\]

The purpose of this paper is to introduce a method for estimating directly \( \{i(n)\} \) and \( \{o(n)\} \) via cepstrum operations on the higher order cumulants of the output data \( \{X(k)\} \). The method is described using third moments (bispectrum) of the data. Its extension to fourth-order cumulants (trispectrum) is shown in Appendix C.

**Lemma I [21], [31]:** Let \( H(z) \) be the system function of a finite-dimensional, linear time-invariant (LTI) exponentially stable system and the system input be a zero-mean non-Gaussian, white, i.i.d. with skewness \( \beta \), then the output bispectrum \( B_z(z_1, z_2) \) exists and is given by

\[
B_z(z_1, z_2) = \beta H(z_1) H(z_2) H(z_1^{-1} z_2^{-1}) \tag{5}
\]

Let us note that the bispectrum of \( \{X_k\} \) is defined as

\[
B_z(z_1, z_2) = \sum_{n=-\infty}^{\infty} R_z(m, n) z_1^{-m} z_2^{-n} \tag{6.1}
\]

where

\[
R_z(m, n) \triangleq \mathbb{E}\{X(k) X(k + m) X(k + n)\} \tag{6.2}
\]

is the third moment sequence.

**Lemma II:** If the system transfer function \( H(z) \) can be written in the form of (3), then the bispectrum of the output sequence exists and is analytic in a region which includes unit surface \( |z_1| = 1, |z_2| = 1 \), i.e.,

\[
B_z(z_1, z_2) = \beta \cdot A(I(z_1^{-1}) I(z_2^{-1}) \cdot 0(z_1) 0(z_2) 0(z_1^{-1} z_2^{-1}) \tag{7.1}
\]

and the third moment sequence \( R_z(m, n) \) is absolutely summable and can be obtained by the inversion formula

\[
R_z(m, n) = \left( \frac{1}{2\pi} \right)^2 \oint_{C_2} \oint_{C_1} B_z(z_1, z_2) \cdot z_1^{-m-1} z_2^{-n-1} dz_1 dz_2 \tag{7.2}
\]

where \( C_1 \) and \( C_2 \) are closed contours which lie completely within the region of convergence of \( B_z(z_1, z_2) \) and encircle the origin counter-clockwise in the plane of the respective variables. The proof is straightforward.
III. THE COMPLEX CEPSTRUM OF THIRD-ORDER MOMENTS

As a 2-D sequence, the third moment sequence \( R_c(m, n) \) of the system output data \{ \( X(k) \) \} belongs to the class of functions whose Z-transform is a rational polynomial.

Lemma III [32]: Any 2-D array having a rational Z-transform will also have a well-defined 2-D complex cepstrum provided: i) its Fourier transform is not equal to zero or infinity at any frequency, and ii) any linear phase trend because of time shift is eliminated.

It is apparent that both requirements are well satisfied for (7.1). This is because \(| a_i | < 1 \), \(| b_i | < 1 \), and \(| c_i | < 1 \) for all \( j \), and because third moments suppress all linear phase trends [12]. Homomorphic systems can be well defined even when poles or zeros are on the unit surface. The reason we require here the region of convergence to include the unit surface is to obtain stable cepstrum sequence, and to be able to compute it via least-squares solution or 2-D FFT operations. Therefore, the Z-transform of the 2-D complex cepstrum of third moments \( \{ R_c(m, n) \} \) can be defined as

\[
C_c(z_1, z_2) \triangleq \ln B_c(z_1, z_2)
= \ln |\beta \cdot A| + \ln |I(z_1^{-1})| + \ln |I(z_2^{-1})| + \ln 0(z_1) + \ln 0(z_2) + \ln 0(z_1^{-1}z_2^{-1}).
\]  

(8.1)

Combining (3) and (8), it follows that

\[
C_c(z_1, z_2) = \ln |\beta \cdot A| + \sum_{i=1}^{L_2} \ln (1 - a_i z_1^{-1})
+ \sum_{i=1}^{L_2} \ln (1 - a_i z_2^{-1})
+ \sum_{i=1}^{L_3} \ln (1 - a_i z_1 z_2^{-1})
+ \sum_{i=1}^{L_3} \ln (1 - a_i z_2 z_1^{-1})
- \sum_{i=1}^{L_3} \ln (1 - c_i z_1^{-1})
- \sum_{i=1}^{L_3} \ln (1 - c_i z_2^{-1})
- \sum_{i=1}^{L_3} \ln (1 - c_i z_1 z_2^{-1}).
\]  

(8.2)

In Appendix A we show that we can use the power series expansion or inversion formula similar to (7.2) to obtain the complex cepstrum, \( c_c(m, n) \), of third moments, viz.,

\[
c_c(m, n) \triangleq z^{-m} C_c(z_1, z_2)
\]

or

\[
c_c(m, n) = \begin{cases} 
\ln |\beta \cdot A| & m = 0, \ n = 0 \\
\frac{1}{n} & m = 0, \ n > 0 \\
\frac{1}{m} & m = 0, \ n > 0 \\
\frac{1}{m} B^{(-m)} & n = 0, \ m < 0 \\
\frac{1}{n} B^{(-n)} & m = 0, \ n < 0 \\
\frac{1}{n} B^{(n)} & m = n > 0 \\
\frac{1}{n} A^{(-n)} & m = n < 0 \\
0 & \text{otherwise}
\end{cases}
\]

(9.1)

where

\[
A^{(k)} \triangleq \sum_{i=1}^{L_2} a_i^k - \sum_{i=1}^{L_3} c_i^k
\]

(9.2)

are parameters which contain the minimum and maximum phase information, respectively. Fig. 1 illustrates the cepstrum sequence \( c_c(m, n) \) of (9).

Since \( C_c(z_1, z_2) \) is analytic in its region of convergence which contains the unit surface, we can perform partial differentiation with respect to \( z_1 \) or \( z_2 \) in that region, i.e.,

\[
\frac{\partial C_c(z_1, z_2)}{\partial z_1} = \frac{1}{B_c(z_1, z_2)} \frac{\partial B_c(z_1, z_2)}{\partial z_1}
\]

or

\[
B_c(z_1, z_2) \cdot z_1 \cdot \frac{\partial C_c(z_1, z_2)}{\partial z_1} = z_1 \frac{\partial B_c(z_1, z_2)}{\partial z_1}.
\]

(10)

From Lemma III, Appendix A, and (10), it follows that the third moment sequence \( \{ R_c(m, n) \} \) is related to its complex cepstrum sequence \( \{ c_c(m, n) \} \) via the linear convolution formula

\[
R_c(m, n) * \{-mc_c(m, n)\} = -m \cdot R_c(m, n)
\]

(11.1)

or

\[
\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} k c_c(k, l) R_c(m - k, n - l) = -m R_c(m, n).
\]

(11.2)

Equation (11) is of fundamental importance because it will serve as the basis for our method. It is shown in Appendix B that the use of the differential cepstrum [9], [10] of \( \{ R_c(m, n) \} \), instead of the complex cepstrum, results into the same convolution relationship (11). This implies that
for our method, either complex cepstrum or differential cepstrum may be used and both will yield the same result. The differential cepstrum \{c_d(m, n)\} of \{R_c(m, n)\} given by (B.3) is illustrated in Fig. 2.

IV. THE CEPSTRAL EQUATION

If we substitute (9.1) into (11.2), after some algebra we obtain the following key cepstral equation:

\[
\sum_{k=1}^{n} A^{(k)}[R_c(m - k, n) - R_c(m + k, n + k)] \\
+ B^{(k)}[R_c(m - k, n - k) - R_c(m + k, n)] \\
= -mR_c(m, n) 
\]

(12)

which provides a direct relationship between parameters \{A^{(k)}\}, \{B^{(k)}\}, and third moments \{R_c(m, n)\}.

A. Connection with the Differential Cepstrum of Impulse Response

From the system transfer function \(H(z)\) in (3) and its impulse response components in (4), it follows that the system impulse response \(\{h(n)\}\) is given by

\[
h(n) = i(n) * o(n). \tag{13}
\]

The differential cepstrum of \(h(n)\) is defined in [9] and [10] as

\[
h_d(n) \overset{\Delta}{=} Z^{-1}\left\{ \frac{1}{H(z)} \frac{\partial H(z)}{\partial \varepsilon} \right\} \tag{14}
\]

and shown to be

\[
h_d(n) = i_d(n) + o_d(n) \tag{15}
\]

where \(\{i_d(n)\}\), \(\{o_d(n)\}\) are the differential cepstra of \(\{i(n)\}\) and \(\{o(n)\}\), respectively. From [9], [10], (9), and (15), we obtain the following elegant result:

\[
i_d(n) = \begin{cases} 
A^{(n-1)} & n \geq 2 \\
0 & n \leq 1
\end{cases} \tag{16.1}
\]

\[
o_d(n) = \begin{cases} 
0 & n \geq 1 \\
b^{(n-1)} & n \leq 0
\end{cases} \tag{16.2}
\]

which implies that the parameters \(\{A^{(k)}\}\), \(\{B^{(k)}\}\) defined by (9.2) and shown in (12) correspond to differential cepstrum coefficients of the system impulse response.

Equations (13)–(16) would allow us to estimate the impulse response of the system via its relationships with differential cepstra. It has been shown in [9] and [10] that the following recursive relations hold:

\[
i(n) = -\frac{1}{n} \sum_{k=2}^{n+1} i_d(k) i(n - k + 1) \tag{17.1}
\]

for \(n \geq 1\)

\[
o(n) = -\frac{1}{n} \sum_{k=2}^{n+1} o_d(k) o(n - k + 1) \tag{17.2}
\]

for \(n \leq -1\)

\[
i(0) = o(0) = 1.
\]

The system impulse response follows from (13). It should be pointed out that \(i(n) \to 0\) as \(n \to \infty\), as well as \(o(n) \to 0\) as \(n \to -\infty\).

It is important to note that the complex cepstrum of \(h(n)\) is defined as [5], [8]

\[
h_c(n) \overset{\Delta}{=} Z^{-1}\{\ln H(z)\} \tag{18}
\]

and is related to the differential cepstrum [9], [10] by

\[
h_c(n) = -\frac{1}{n} h_d(n + 1) \quad (n \neq 0). \tag{19}
\]

B. Properties

Several properties are associated with the nonminimum phase signal reconstruction approach described in this section.
1) The minimum and maximum phase components can be reconstructed separately. This is achieved by (17.1) and (17.2).

2) The method will work for MA, AR, or ARMA signals. Sufficiently long impulse responses can be reconstructed by this method to serve as good approximations of AR or ARMA impulse responses.

3) It will not require model order selection criteria because it will reconstruct the system impulse response directly bypassing estimation of the model parameters.

4) The approach does not require any a priori knowledge of the type of the system (AR, MA, or ARMA).

5) The length of minimum and maximum impulse responses may be determined by the algorithm itself. In other words, recursions (17.1) and (17.2) could be terminated when successive values of \(|i(n)|\) and \(|o(n)|\) stay below a threshold value \(\gamma\), where \(\gamma\) is very small (depending on word length).

The extension of the method to the fourth-order cumulant domain (trispectrum) is somewhat straightforward and is given in Appendix C. It can be useful when the skewness of the input non-Gaussian white noise \(W(k)\) is very close to, or identical to, zero.

V. THE RECONSTRUCTION METHODS (BICEPSTRUM)

The objective is to reconstruct the minimum \(\{i(n)\}\) and maximum \(\{o(n)\}\) phase impulse response components from the third moments of the input data. We show in this section how to achieve such an objective by utilizing either the cepstral equation (12) or the convolution formula (11.1). Once the cepstrum coefficients \(\{A^{(k)}\}\) and \(\{B^{(k)}\}\) are estimated, then equations (16) and (17) provide us a way to reconstruct the impulse response of the system.

A. Least-Squares (LS) Estimation of \(\{A^{(k)}\}\), \(\{B^{(k)}\}\)

From the definition of \(\{A^{(k)}\}\), \(\{B^{(k)}\}\) in (9.2), and the original assumptions that \(|a_i| < 1\), \(|b_i| < 1\), and \(|c_i| < 1\) for all \(i\), it follows that in practice we can always truncate (12) arbitrarily closely and obtain the approximate cepstral equation

\[
\begin{align*}
\sum_{i=1}^{p} \left( A^{(i)} R_e(m - i, n) - R_e(m + i, n + i) \right) \\
+ \sum_{j=1}^{q} \left( B^{(j)} R_e(m - j, n - j) - R_e(m + j, n) \right) \\
= -m R_e(m, n). 
\end{align*}
\]

(20)

This is a reasonable approximation because parameters \(\{A^{(k)}\}\), \(\{B^{(k)}\}\) decay exponentially. The \(p\), \(q\) are integers and may be chosen using

\[
\begin{align*}
p &= \ln c / \ln a \\
q &= \ln c / \ln b
\end{align*}
\]

(21.1)

where \(\max \{|a_i|, |c_i|\} < a < 1\)

\[
\max \{|b_j|\} < b < 1 \tag{21.2}
\]

and \(c\) is a very small constant (say, \(10^{-4}\)). So, \(A^{(k)} = 0\) for \(k > p\) and \(B^{(k)} = 0\) for \(k > q\). It is therefore apparent that this approach requires some a priori knowledge about the pole-zero location of the system's transfer function, i.e., the values (a) and (b) of (21.2).

Assuming that the third moments \(\{R_e(m, n)\}\) are available (in practice their estimates), we can define \(w \triangleq \max \{p, q\}\) and \(z \triangleq \lfloor w/2 \rfloor\) integers and choose \(m = -w, \ldots , 0, \ldots , w\) and \(n = -z, \ldots , 0, \ldots , z\) to form the overdetermined system of equations

\[
Ya = y \tag{22}
\]

where \(Y = [(2w + 1) \times (2z + 1)] \times (p + q)\) matrix whose entries are expressions of the form \(\{R_e(m, n)\} - R_e(m, n)\) (see 20).

\[
a = [A^{(1)}, \ldots , A^{(p)}, B^{(1)}, \ldots , B^{(q)}]^{T} \text{is a}(p + q) \times 1 \text{unknown set of parameters, and } y = [(2w + 1) \times (2z + 1)] \times 1 \text{vector whose entries are expressions of the form } \{-mR_e(m, n)\} \text{ (see 20).}
\]

For example, if \(p = q = 20\), then \(w = 20, z = 10\), and \(Y\) is \(1681 \times 40\) matrix.

The least-squares solution to (22) is

\[
a = [Y^{T}Y]^{-1}Y^{T}y. \tag{23}
\]

One of the advantages of getting parameters \(\{A^{(k)}\}\), \(\{B^{(k)}\}\) directly from third moments \(\{R_e(m, n)\}\) (or their estimates) via (23) is that we avoid computations associated with (6.1), (8.1), and (9) which require bispectrum estimation procedures as well as phase unwrapping algorithms. The limitation, however, is that if poles or zeros come close to the unit circle, the computational cost of (23) will rise.

B. FFT-Based Estimation of \(\{A^{(k)}\}\), \(\{B^{(k)}\}\)

An alternative method for computing the cepstrum coefficients in (9.1) that does not require phase unwrapping can be based on two-dimensional (2-D) fast Fourier transform (FFT) operations. From (10) or (11.1) we obtain

\[
m \cdot c_i(m, n) = F^{-1}\left\{F[m \cdot R_e(m, n)]/F[R_e(m, n)]\right\} \tag{24}
\]

where \(F\{\cdot\}\) represents the 2-D Fourier transform. Therefore, \(c_i(m, n)\) can be computed using either conventional (2-D FFT computations) or parametric bispectrum estimation algorithms [12]. We choose in this paper to implement the direct conventional method with additional averaging in the frequency domain (see [12] for details).

The big computational advantage of using this estimation approach over the least-squares (LS) one is for systems with pronounced resonances or antiresonances, i.e., when \(p\) and \(q\) are very large.
VI. THE ALGORITHMS

Let us assume that \( \{X(k), k = 1, 2, \cdots, L\} \) is the output sequence of (1). Then we have the following.

1) Estimate the third moment sequence \( R_3(m, n) \) of the data as follows [12].

(i) Segment the data into \( N_L \) records of \( M \) samples each, i.e.,

\[
Y_j^{(i)} = X(j + (i - 1)M)
\]

\( i = 1, 2, \cdots, N_L, \quad j = 1, 2, \cdots, M. \) \hfill (25.1)

(ii) Generate

\[
r_k^{(i)}(m, n) = \frac{1}{M} \sum_{j=1}^{S_i} Y_j^{(i)} Y_{j+m}^{(i)} Y_{j+n}^{(i)}
\]

\( i = 1, 2, \cdots, N_L, \)

\( S_i = \max \{ 1, -m, -n \}, \)

\( S_2 = \min \{ M, M - m, M - n \}. \) \hfill (25.2)

(iii) Average over all segments

\[
\hat{R}_3(m, n) = \frac{1}{N_L} \sum_{i=1}^{N_L} r_k^{(i)}(m, n).
\]

\hfill (25.3)

(iv) Use a window function

\[
\hat{R}_3(m, n) = \hat{R}_3(m, n) \cdot W(m, n).
\]

\hfill (25.4)

See [12] for various types of windows.

2a) Least-Squares (LS) Method: Replace \( R_3(m, n) \) in (20) by \( \hat{R}_3(m, n) \). Choose \( p, q \) using (21). Form the overdetermined system of equations (22) and obtain LS solution (23), i.e.,

\[
\hat{a} = [\hat{A}^{(1)}, \cdots, \hat{A}^{(p)}, \hat{B}^{(1)}, \cdots, \hat{B}^{(q)}]^T.
\]

\hfill (26)

2b) FFT-Based Method: Given \( \{ \hat{R}_3(m, n) \} \), \( m, n = -J_1, \cdots, 0, \cdots, J_1 \), from (25.4), compute

\[
B_1(\lambda_1, \lambda_2) = \sum_{m=-J_1}^{J_1} \sum_{n=-J_1}^{J_1} [m \hat{R}_3(m, n)]
\]

\[
\cdot \exp -j \frac{2\pi}{J} [\lambda_1 m + \lambda_2 n]
\]

\hfill (27.1)

\[
B_2(\lambda_1, \lambda_2) = \sum_{m=-J_1}^{J_1} \sum_{n=-J_1}^{J_1} \hat{R}_3(m, n)
\]

\[
\cdot \exp -j \frac{2\pi}{J} [\lambda_1 m + \lambda_2 n]
\]

\hfill (27.2)

and

\[
B(\lambda_1, \lambda_2) = B_1(\lambda_1, \lambda_2)/B_2(\lambda_1, \lambda_2)
\]

\hfill (27.3)

where

\[
\lambda_1, \lambda_2 = -J/2, \cdots, 0, \cdots, J/2 \quad (J \gg 2J_1 + 1).
\]

By averaging in the frequency domain in a rectangle of size \( (2L_1 + 1) \times (2L_1 + 1) \) to reduce variance, we obtain

\[
\hat{B}(k_1, k_2) = \sum_{l_1 = -L_1}^{L_1} \sum_{l_2 = -L_2}^{L_2} B[k_1(2L_1 + 1)
\]

\[+ l_1, k_2(2L_2 + 1) + l_2]

\hfill (27.4)

where

\[
k_1, k_2 = -N_o/2, \cdots, 0, \cdots, N_o/2
\]

and \( N_o \) is the closest integer to \( J/(2L_1 + 1) \). For example, if \( J = 512 \) and \( L_1 = 2 \), then \( N_o = 102 \).

The cepstrum coefficients are computed using

\[
m\hat{c}(m, n) = \frac{1}{N_o^2} \sum_{k_1 = -N_o/2}^{N_o/2} \sum_{k_2 = -N_o/2}^{N_o/2} \hat{B}(k_1, k_2)
\]

\[
\cdot \exp -j \frac{2\pi}{N_o} [k_m + k_n]
\]

\hfill (27.5)

and the \( \{ A^{(k)} \}, \{ B^{(k)} \} \) follows from (9.1).

The parameter \( J_1 \) should be chosen sufficiently large so that \( \hat{R}_3(m, n) = 0 \) for \( |m|, |n| > J_1 \). One rule of thumb for choosing \( J_1 \) is to take it greater than or equal to the length of the autocorrelation sequence of the data. On the other hand, parameter \( N_o \) should be chosen so that \( N_o \geq 2 \max \{ p, q \} \).

3) Using (16), form the identities

\[
i_d(n) = \begin{cases} \hat{A}^{(n-1)} & 2 \leq n \leq p + 1 \\ 0 & \text{otherwise} \end{cases}
\]

\hfill (28.1)

and

\[
\delta_d(n) = \begin{cases} 0 & \text{otherwise} \end{cases}
\]

\hfill (28.2)

4) Initialize \( i(0) = \delta(0) = 1 \). Start recursions (17) with estimated components, i.e.,

\[
i(n) = -\frac{1}{n} \sum_{k=2}^{n+1} i_d(k) \hat{i}(n - k + 1),
\]

\( n = 1, 2, 3, \cdots, \)

\hfill (29.1)

\[
\delta(n) = -\frac{1}{n+1} \sum_{k=2}^{n+1} \delta_d(k) \hat{\delta}(n - k + 1),
\]

\( n = -1, -2, -3, \cdots. \)

\hfill (29.2)

Let us note that while \( \hat{i}(n) \) is a causal sequence, \( \delta(n) \) is an anticausal sequence. Equations (29.1) and (29.2) may also be performed using FFT algorithms.

5) Terminate recursion (29.1) at \( n = N_o \), where \( |\hat{i}(n)| < \gamma \) (very small) for all \( n > N_1 \). Similarly, terminate recursion (29.2) at \( n = -N_o \) where \( |\delta(n)| < \gamma \) for all \( n < -N_2 \). The minimum phase impulse response is \( \{ \hat{i}(1), \hat{i}(2), \cdots, \hat{i}(N_o) \} \) and the maximum phase \( \{ \hat{\delta}(1), \hat{\delta}(2), \cdots, \hat{\delta}(N_o) \} \).
6) Perform Fourier transform

\[ \hat{I}(\omega) = \sum_{n=0}^{N_1} \hat{i}(n) e^{-j\omega n} \]

\[ \hat{\delta}(\omega) = \sum_{n=-N_2}^{0} \hat{\delta}(n) e^{-j\omega n}. \]

From (3.1) it follows that the system frequency transfer function is given by

\[ \hat{H}(\omega) = \hat{I}(\omega) \cdot \hat{\delta}(\omega). \]

VII. SIMULATION RESULTS

In this section we present computer simulation results we have conducted to investigate the performance of the methods introduced in this paper and compare them to conventional methods. We illustrate examples in which the system is minimum phase moving average (MA), mixed phase MA, as well as mixed phase ARMA. The system driving noise \( W(k) \) is zero-mean non-Gaussian (exponentially distributed, white, and was generated by the GGENX subroutine of the IMSL library with skewness \( \beta = E(W^3) = 2 \). Eight different lengths of output data have been used for each simulation example: \( 1 \times 128 \) (one segment of data with 128 samples per segment), \( 2 \times 128, 4 \times 128, 8 \times 128, 16 \times 128, 32 \times 128, 64 \times 128, \) and \( 128 \times 128 \). Sample mean and standard deviation of magnitude and phase response at each frequency have been computed using 100 different output data realizations of the same statistical description. The results are compared to the true magnitude and phase response which are computed using the true third moment sequence of the system output in each example.

A. Minimum Phase MA System

The results obtained using a second-order MA system are illustrated in Table 1. Specifically, the estimated cepstral parameters \( \{ A(k) \} \) via least-squares method as well as the resulting minimum phase \( \{ \hat{r}(n) \} \) and maximum phase \( \{ \hat{\delta}(n) \} \) impulse response sequences, are illustrated in terms of sample mean and standard deviation for the various output record lengths and compared to the true ones. Fig. 3 shows the true magnitude and phase responses as well as their estimated means and standard deviations for each frequency. From Table I and Fig. 3, it is apparent that bias and variance in the estimates decrease as the length of the data increases. It is also apparent that the bicepsal LS method works very well even at “very short” length data records \((2 \times 128)\). The values of \( p, q \) were selected \( p = q = 4 \).

B. Mixed Phase MA System

Table II illustrates the results obtained by the bicepsal LS method \((p = q = 4)\) using a second-order mixed phase
Fig. 3. (a) Minimum phase MA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are $1 \times 128$ and $2 \times 128$. (b) Minimum phase MA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are $8 \times 128$ and $16 \times 128$. 
TABLE II
MIXED PHASE MA SYSTEM IDENTIFIED VIA CEPSRUM OF THIRD MOMENTS
(LEAST-SQUARES METHOD) \((p = q = 4) H(z) = (1-0.5z^{-1})(1-0.35z^{-1})\)

<table>
<thead>
<tr>
<th>DATA NUMBER</th>
<th>TRUE VALUES</th>
<th>128*128</th>
<th>64*128</th>
<th>32*128</th>
<th>16*128</th>
<th>8*128</th>
<th>4*128</th>
<th>2*128</th>
<th>1*128</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td></td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
<td>m &amp; s</td>
</tr>
<tr>
<td>1</td>
<td>.3496</td>
<td>.3646</td>
<td>.0002</td>
<td>.3677</td>
<td>.0005</td>
<td>.3467</td>
<td>.0010</td>
<td>.3489</td>
<td>.0019</td>
</tr>
<tr>
<td>2</td>
<td>.1219</td>
<td>1.174</td>
<td>.0004</td>
<td>.1212</td>
<td>.0009</td>
<td>.1153</td>
<td>.0024</td>
<td>.1293</td>
<td>.0035</td>
</tr>
<tr>
<td>3</td>
<td>.0413</td>
<td>.0420</td>
<td>.0017</td>
<td>.0448</td>
<td>.0031</td>
<td>.0541</td>
<td>.0069</td>
<td>.0624</td>
<td>.0146</td>
</tr>
<tr>
<td>4</td>
<td>.0127</td>
<td>.0027</td>
<td>.0028</td>
<td>.0106</td>
<td>.0061</td>
<td>.0171</td>
<td>.0094</td>
<td>.0297</td>
<td>.0143</td>
</tr>
<tr>
<td>(B)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.4964</td>
<td>.4974</td>
<td>.0002</td>
<td>.4965</td>
<td>.0004</td>
<td>.4987</td>
<td>.0008</td>
<td>.4981</td>
<td>.0001</td>
</tr>
<tr>
<td>2</td>
<td>.2490</td>
<td>.2423</td>
<td>.0003</td>
<td>.2416</td>
<td>.0015</td>
<td>.2420</td>
<td>.0026</td>
<td>.2438</td>
<td>.0035</td>
</tr>
<tr>
<td>3</td>
<td>.1214</td>
<td>.1228</td>
<td>.0002</td>
<td>.1171</td>
<td>.0025</td>
<td>.1280</td>
<td>.0063</td>
<td>.1404</td>
<td>.0127</td>
</tr>
</tbody>
</table>

\(\bar{m}\) --- mean  
\(\bar{C}\) --- standard deviation

MA system. In addition, Fig. 4 shows the true as well as estimated magnitude and phase response of the system. Clearly the results are very consistent. However, comparing Fig. 3 to Fig. 4, it appears that longer data records are required in the mixed phase case to achieve the bias and variance of the minimum phase case. From Table II, it appears that the bicestral LS method works well at 4 \(\times\) 128 record length.

C. Mixed Phase ARMA System

This is an example of an ARMA system with one pole and one zero inside the unit circle, and two zeros outside. Table III and Fig. 5 illustrate the results which clearly demonstrate the effectiveness of the bicestral LS method \((p = q = 4)\) introduced in this paper. Very good quality estimates are obtained even at 4 \(\times\) 128 record length.

D. Mixed Phase MA System with Pronounced Resonances

Fig. 6 illustrates the zero location and the nonminimum phase impulse response of the MA(3, 3) system, and Fig. 7 shows comparisons between the estimated and true magnitude and phase response of the system for various output record lengths. The bicestral LS method was used in this example with \(p = q = 20\). From Fig. 7, it appears that the bias and variance of the phase response decrease as the length of the data increases and become small at 128 \(\times\) 128.

E. Mixed Phase ARMA System with Pronounced Resonances

This is an example of an ARMA system with three poles and one zero inside the unit circle, and two zeros outside. Fig. 8 illustrates the actual pole-zero location of the system as well as its impulse response, whereas Fig. 9 presents the true and estimated (via bicestral LS method) magnitude and phase responses for four different record lengths. It is apparent that small bias and variance estimates are obtained even at 8 \(\times\) 128 record length. For this example, \(p = 25\) and \(q = 15\) were selected.

F. Unbiased versus Biased Third Moment Estimates

Biased third moment estimates obtained by (25) have been used in all examples presented in this paper. Unbiased estimates were also obtained [via minor modification of (25.2)] in these examples and we have found that no difference is made. One of these results is illustrated in Fig. 10 for the MA system described in Subsection D, and record length 64 \(\times\) 128.

G. FFT-Based Bicestral Method

Fig. 11(a) and (b) illustrates the true and estimated magnitude and phase response of the mixed phase MA system described in Subsection B. The FFT-based bicestral method has been used here to estimate the \(\{A^{(4)}\}\) and \(\{B^{(4)}\}\) cepstrum coefficients. Fig. 11(a) corresponds
Fig. 4. (a) Mixed phase MA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are 4 x 128 and 8 x 128. (b) Mixed phase MA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are 16 x 128 and 64 x 128.
to $N_o = 128$ and $L_1 = 0$, whereas Fig. 11(b) corresponds to $N_o = 128$ and $L_1 = 2$ (i.e., averaging in the frequency domain over a $5 \times 5$ window). Clearly, the averaging in the frequency domain significantly reduces bias and variance in the estimates. Comparing Fig. 11(b) to Fig. 4 (bisecktron LS method), it is apparent that less variance is achieved with the LS method for this specific example. However, comparing Fig. 11(c) to Fig. 7 (different example), we see that LS and FFT-based methods give sensibly identical results.

**H. Comparison to Conventional Methods**

We illustrate in Fig. 12 phase response estimates obtained by the Matsuoka–Ulyrich (MU) conventional method which is based on conventional bispectrum estimation techniques [12], [13]. The method was applied on exactly the same data used by the bisecktron LS method in Subsections A, B, and C. Comparing the results shown by Figs. 3–5 to those illustrated by Fig. 12, it is apparent that less in value variance estimates are generated by the bisecktron LS approach.

**I. Computational Complexity**

Table IV summarizes computational complexity comparisons between conventional and bisecktron methods. The figure of merit used is the number of multiplications (MUL) (log_{10} MUL). It is also assumed that the data are segmented into records of 128 samples each. The conventional approach employs the Yule–Walker (YW) method with autocorrelations for magnitude response reconstruction and the Matsuoka–Ulyrich (MU) method for phase reconstruction [12], [13]. From this table, it appears that the conventional approach is generally faster than bisecktron techniques. Within the bisecktron family of techniques, the LS method appears to require fewer multiplications than the FFT-based methods for even $p + q = 40$.

**VIII. Conclusions**

A new method has been presented for nonminimum phase system identification based on the idea of computing the complex cepstrum of the higher order cumulants of the system output. The method is flexible enough to reconstruct the magnitude and phase response of MA, AR, or ARMA systems. It may be computationally efficient and does not require a priori knowledge of the type of the system. It has been demonstrated by means of standard examples that the method works very well and provides estimates with smaller bias and variance than conventional methods, especially with “short” length data records.
Fig. 5. (a) Mixed phase ARMA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are 1×128 and 4×128. (b) Mixed phase ARMA system identified via cepstrum of third moments using least-squares method ($p = q = 4$). Lengths of data are 8×128 and 16×128.
MA (3, 3)

\[ h(z) = (1 - 0.869z^{-1})(1 + 0.617z^{-2})(1 - 0.55z^{-1})(1 - 1.2z^{-1} + 0.45z^{-2}) \]

Fig. 6. Mixed MA system with pronounced resonances: (a) zero location, (b) impulse response, and (c) \((A^*)\), \((B^*)\) cepstrum coefficients.

\[ h(z) = (1 - 0.869z^{-1})(1 + 1.1z + 0.617z^{-2})(1 - 0.55z^{-1})(1 - 1.2z^{-1} + 0.45z^{-2}) \]

--- Estimated

--- True

Fig. 7. (a) Mixed phase MA system with pronounced resonances identified via cepstrum of third moments using least-squares method \((p = q = 20)\). Lengths of data are 8 \(\times\) 128 and 16 \(\times\) 128.
\[ H(z) = (1 - 0.8692z^{-1}) (1 + 1.12 + 0.6172z^2) (1 - 0.852z^{-1}) (1 - 1.2z^{-1} + 0.45z^{-2}) \]

(b) Mixed phase MA system with pronounced resonances identified via cepstrum of third moments using least-squares method \((p = q = 20)\). Lengths of data are 32 \times 128 and 128 \times 128.

AR(1), MA(1,2)

\[ H(z) = (1 - 0.92z^{-1}) (1 - 0.52 + 0.4172z^2) (1 - 0.86z^{-1}) (1 - 1.2 + 0.61z^{-2}) \]

(c) Mixed phase ARMA system with pronounced resonances: (a) pole-zero location, (b) impulse response, and (c) \((A^{(k)})\), \((B^{(k)})\) cepstrum coefficients.
Fig. 9. (a) Mixed phase ARMA system with pronounced resonances identified via cepstrum of third moments using least-squares method ($p = 25$, $q = 15$). Lengths of data are $4 \times 128$ and $8 \times 128$. (b) Mixed phase ARMA system with pronounced resonances identified via cepstrum of third moments using least-squares method ($p = 25$, $q = 15$). Lengths of data are $32 \times 128$ and $64 \times 128$. 

\[
H(z) = \frac{(1-0.92z^{-1})(1-1.52z^2+0.6176z^4)}{(1-0.86z^{-1})(1-z^2+0.61z^{-2})}
\]

--- Estimated
--- True

(a)

(b)
Fig. 10. Mixed phase MA system identified via cepstrum of third moments via least-squares method: (a) unbiased third moment estimates, (b) biased third moment estimates.

Fig. 11. Mixed phase MA system identified via cepstrum of third moments using FFT-based method: (a) no averaging in the frequency domain ($L_x = 0$), (b) averaging over a $3 \times 3$ rectangle ($L_x = 2$), and (c) different MA system using FFT-based method with no averaging.
MATSUMO-KAULICH ALGORITHM

$$H(Z) = (1-0.5Z^{-1})(1-0.35Z^{-1})$$

Fig. 12. Phase reconstruction in the bispectrum domain via the Matsumo-
Ulrich (MU) conventional method: (a) mixed phase MA system, (b) 
minimum phase MA system, and (c) mixed phase ARMA system (com-
pare to Figs. 3–5).
### Table IV
Number of Multiplications (log₁₀ MUL) Required by the Conventional and Biceptrum Methods

<table>
<thead>
<tr>
<th>Number of Cepstrum Parameters (p + q)</th>
<th>Number of Records</th>
<th>Conventional Method</th>
<th>Biceptrum Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MU(phase)</td>
<td>YW(magn.)</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>6.455</td>
<td>5.106</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>6.399</td>
<td>5.309</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
<td>6.379</td>
<td>5.720</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6.384</td>
<td>5.759</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>6.399</td>
<td>5.871</td>
</tr>
</tbody>
</table>

In the form of closing remarks, we would like to point out that there are some disadvantages associated with the use of cepstra: i) cepstra are not defined for band-limited signals; ii) the computational complexity of the biceptrum methods (LS or FFT-based) rises when poles or zeros of the system come close to the unit circle; and iii) the biceptrum methods require a prior knowledge or good estimates of p, q parameters. If p, q are to be severely underestimated, the phase and magnitude information will not be recovered accurately.

**Appendix A**

Let \( \hat{a} = \max \{ |a_i|, 1 \leq i \leq L_1, \hat{b} = \max \{ |b_i|, 1 \leq i \leq L_2 \}, \text{ and } \hat{c} = \max \{ |c_i|, 1 \leq i \leq L_3 \} \). Also, \( e = \max \{ \hat{a}, \hat{b}, \hat{c} \} \). Then, the region of convergence for \( C_d(z_1, z_2) \) is \( R_{c_d} = \{ |z_1| > e, |z_2| > e \text{ and } |z_1 z_2| < 1/e \} \). It is easy to see that the unit surface \( \{ |z_1| = 1, |z_2| = 1 \} \) lies within \( R_{c_d} \).

**Appendix B**

For \( B_c(z_1, z_2) \) of (7.1), we can define its differential cepstrum [9], [10] as:

\[
c_d(m, n) = Z^{-1} \left\{ \frac{\partial B_c(z_1, z_2)}{\partial z_1} \right\}
\]

Substituting (3) and (7.1) into (B.1), we obtain:

\[
C_d(z_1, z_2) = \sum_{i=1}^{L_1} \frac{a_i z_1^{-i}}{1 - a_i z_1^{-1}} - \sum_{i=1}^{L_2} \frac{a_i z_2^{-i}}{1 - a_i z_2^{-1}}
- \sum_{i=1}^{L_2} b_i z_1^{-i} + \sum_{i=1}^{L_3} b_i z_1^{-i} z_2^{-i}
- \sum_{i=1}^{L_3} c_i z_1^{-i} + \sum_{i=1}^{L_4} c_i z_1^{-i} z_2^{-i}
\]

\]

and

\[
c_d(m, n) = \begin{cases} A^{(m-1)} & n = 0, m \geq 2 \\ -B^{(1-n)} & n = 0, m \leq 0 \\ -A^{(n)} & n \leq -1, m = n + 1 \\ B^{(n)} & n \geq 1, m = n + 1 \\ 0 & \text{otherwise} \end{cases}
\]

\]

where \( A^{(n)} \) and \( B^{(n)} \) are given by (9.2). From (B.1) we have that:

\[
c_d(z_1, z_2) z_1 B_c(z_1, z_2) = z_1 \frac{\partial B_c(z_1, z_2)}{\partial z_1}
\]
or
\[ c_d(m, n) = R_z(m + 1, n) = -mR_z(m, n). \] (B.4)
If we substitute (9) into (11) and (B.3) into (B.4), we find that (11) is identical to (B.4). Let us also note that differentiation with respect to \( z_2 \) in (B.1) yields the same expression (B.3).

**APPENDIX C**

The output \( \{X(k)\} \) of the ARMA system (1) has trispectrum [21], [28], [31].
\[
T_s(z_1, z_2, z_3) = \gamma I(z_1^{-1}) I(z_2^{-1}) I(z_3^{-1})
\cdot I(z_1 z_2 z_3) 0(z_1) 0(z_2) 0(z_3)
\cdot 0(z_1^{-1} z_2^{-1} z_3^{-1})
\] (C.1)
where
\[
\gamma = E\{W^4(k)\} - 3Q^2
\]
is the fourth-order cumulant of the driving white noise sequence.

Its complex cepstrum is defined as
\[
c_{c}(m, n, l) \triangleq Z^{-l}\{\ln \{T_s(z_1, z_2, z_3)\}\}\] (C.2)
and by Taylor expansion in the similar manner to Appendix B, it takes the form
\[
c_c(m, n, l) = \begin{cases} 
\ln |\gamma| & m = n = l = 0 \\
-\frac{1}{m} A^{(m)} & m > 0, n = l = 0 \\
-\frac{1}{n} A^{(n)} & n > 0, m = l = 0 \\
-\frac{1}{l} A^{(l)} & l > 0, m = n = 0 \\
\frac{1}{m} B^{(-m)} & m < 0, n = l = 0 \\
\frac{1}{n} B^{(-n)} & n < 0, m = l = 0 \\
\frac{1}{l} B^{(-l)} & l < 0, m = n = 0 \\
-\frac{1}{n} B^{(n)} & m = n = l > 0 \\
\frac{1}{n} A^{(n)} & m = n = l < 0 \\
0 & \text{otherwise}
\end{cases}
\]
where \( \{A^{(k)}\}, \{B^{(k)}\} \) are given by (9.2).

It is also straightforward to verify that the following identity holds between the fourth-order cumulant sequence, \( F_s(m, n, l) \), and its complex cepstrum \( c_c(m, n, l) \):
\[
F_s(m, n, l) = -m F_s(m, n, l) = -m F_s(m, n, l).
\] (C.4)

Let us note that
\[
F_s(m, n, l)
\]
\[
\triangleq \text{Cum} \{X(k) X(k + m) X(k + n) X(k + l)\}
\]
\[
= E\{X(k) X(k + m) X(k + n)
\cdot X(k + l)\} - \{r(m) \cdot r(l - n) + r(n) \cdot r(l - m) + r(l) \cdot r(n - m)\},
\] (C.5)
where
\[
r(n) \triangleq E\{X(k) X(k + n)\}. \] (C.6)
Substituting (C.3) into (C.4), we obtain the trispectrum cepstral equation
\[
\sum_{k=1}^{\infty} \left[ A^{(k)} \right] F_s(m - k, n - l) - F_s(m + k, n + k, l + k)
+ B^{(k)} \left[ F_s(m - k, n - k, l - k) - F_s(m + k, n, l) \right] = -m F_s(m, n, l). \] (C.7)

Equation (C.7) is the key cepstral equation that can be approximated and solved for \( \{A^{(k)}\}, \{B^{(k)}\} \) via least squares. The procedures to follow are entirely analogous to those of the bispectrum described in Section V.

**ACKNOWLEDGMENT**

The authors are much indebted to two anonymous reviewers for their comments and constructive criticisms.

**REFERENCES**


Renlong Pan (S’87) was born in Shanghai, China, on October 20, 1940. He received the B.S. degree in instrumentation from Harbin Institute of Technology, China, in 1962, and the M.S. degree in control from Shanghai University of Technology, China, in 1981.

During 1981-1984 he was with the Department of Automation, Shanghai University of Technology, as a Lecturer. He is currently a Research Assistant in the Department of Electrical and Computer Engineering, Northeastern University, working toward the Ph.D. degree. His main research interests are higher order spectrum and its application in geophysics and communication, sensor array processing, and radar/sonar systems.

Chrysostomos L. Nikias (S’76–M’82–SM’87) received the diploma in electrical and mechanical engineering from the National Technical University of Athens, Greece, in 1977, and the M.S. and Ph.D. degrees in electrical engineering from the State University of New York at Buffalo, in 1980 and 1982, respectively.

He is Associate Professor and Director of the Communications and Digital Signal Processing (CDSIP) Center for Research and Graduate Studies, Department of Electrical and Computer Engineering, Northeastern University, Boston, MA. From 1982 to 1985 he was on the Faculty of the University of Connecticut, Storrs. His research interests lie in the areas of digital signal processing with applications, detection and estimation theory, and bioengineering. He consults for the Maryland Institute for Emergency Medical Services Systems (MIEMSS), Baltimore, MD; ALCOA, Pittsburgh, PA; and the Gordon Institute, Wakefield, MA. He has organized and taught extensively short courses devoted to continuing education engineering. He is the designer and instructor of a videotape seminar (15 h) entitled “Modern Spectrum Estimation and Array Processing,” Network Northeastern, Boston, MA, 1987.

Dr. Nikias is a member of Tau Beta Pi and the Technical Chamber of Greece. He is also a member of the Conference Board of the IEEE Acoustics, Speech, and Signal Processing (ASSP) Society, and the ASSP Technical Committee on Spectrum Estimation and Modeling. He has previously served as Associate Editor of the IEEE TRANSACTIONS ON ACoustics, Speech, and Signal Processing (1985–1987), and as the Co-Chairman of the Third ASSP Workshop on Spectrum Estimation and Modeling (1986).