

# Outage Optimality of Opportunistic Amplify-and-Forward Relaying

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**Abstract**—In this paper, we show that optimal selection and transmission of a single relay among a set of multiple amplify-and-forward (AF) candidates minimize the outage probability (i.e., outage-optimal) and outperform any other strategy that involves simultaneous transmissions from more than one AF relay under an aggregate power constraint. This outage optimality demonstrates that cooperation benefits are maximized with intelligent scheduling among AF relays.

**Index Terms**—Cooperative diversity, fading relay channel, outage probability, wireless networks.

## I. INTRODUCTION

COOPERATIVE relaying techniques have been proposed as an efficient way to mitigate fading in slow fading wireless environments and have attracted increasing interest [1]–[3]. Although cooperative relaying has focused on simultaneous and in-band transmissions from multiple relays (see, e.g., [3] and references therein), recent asymptotic analysis showed that carefully selected single-relay transmissions incur no performance loss compared to multiple-relay transmissions in terms of the diversity-multiplexing gain tradeoff for both decode-and-forward (DF) and amplify-and-forward (AF) relays [4]. Subsequent finite signal-to-noise ratio (SNR) analysis showed that under an aggregate power constraint, specific single-relay selection among DF relays is globally outage-optimal, i.e., minimizes the outage probability and outperforms techniques based on multiple DF-relay transmissions [5]. Moreover, the optimal rule for single-relay AF transmission was presented—i.e., the *opportunistic* relay selection that minimizes the outage probability among all *single*-relay selection rules in AF environments.

In this letter, we show that the optimal rule for single AF transmission in dual-hop Rayleigh Fading channels is also *globally* outage-optimal among all possible techniques that rely on multiple transmissions of AF relays under an aggregate power constraint. In Section II we present the system and protocol assumptions, and in Section III we formulate

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the problem. A theorem is established to show the outage optimality of single opportunistic AF relaying in Section IV, and its implications are discussed in Section V.

## II. MODELS AND PROTOCOLS

We consider a dual-hop scenario where a single source communicates with a single destination through a total number of  $K$  relays. In line with prior art, the received signal in a link ( $A \rightarrow B$ ) between two nodes A and B is given by  $y_B = \alpha_{AB} x_A + n_B$ , where  $x_A$  is the signal transmitted at the node A,  $\alpha_{AB} \sim \mathcal{CN}(0, \Omega_{AB})$  is the channel gain between the link  $A \rightarrow B$ , and  $n_B \sim \mathcal{CN}(0, N_0)$  is the additive white Gaussian noise (AWGN) at the node B.<sup>1</sup> For each link, let  $\gamma_{AB} \triangleq |\alpha_{AB}|^2$  be the instantaneous squared channel strength, obeying an exponential distribution with  $\mathbb{E}\{\gamma_{AB}\} \triangleq \Omega_{AB}$ . We further assume that the channel gains for all links are statistically independent and that all terminals have the same noise variance  $N_0$ .

For each relay  $k \in \mathcal{S}_{\text{relay}} = \{1, 2, \dots, K\}$ , we designate a link from the source to the  $k$ th relay by  $S \rightarrow k$  and a link from the  $k$ th relay to the destination by  $k \rightarrow D$ . If the node A is the source, then  $\mathbb{E}\{|x_A|^2\} = \mathcal{P}_S$ , while if the node A is the  $k$ th relay, then  $\mathbb{E}\{|x_A|^2\} = v_k \mathcal{P}_R$  is the *individual* relay power, where  $v_k \in [0, 1]$ ,  $\sum_{k=1}^K v_k = 1$ , and  $\mathcal{P}_R$  is the *aggregate* relay power. We also define  $\eta_{S_k} \triangleq \Omega_{S_k} \mathcal{P}_S / N_0$  and  $\eta_{kD} \triangleq \Omega_{kD} \mathcal{P}_R / N_0$ , while  $R$  denotes the end-to-end spectral efficiency in bps/Hz.

We consider a two-phase communication protocol where the source can communicate with the destination only through half-duplex relays (no direct path between the source and destination). During the first phase, the source (with no channel knowledge) transmits  $N/2$  symbols and all relays listen, while during the second phase, the relays forward a scaled version  $\sqrt{v_k \mathcal{P}_R} y_k / \sqrt{\mathbb{E}\{|y_k|^2\}}$  of their received signal  $y_k$  (acquired during the first phase of the protocol), using the same number of symbols. The channel is assumed to remain constant during the two phases (at least  $N$ -symbol coherence time).

## III. PROBLEM FORMULATION

The mutual information for multiple AF transmissions with  $K$  relays under the aggregate power constraints (with no direct

<sup>1</sup> $\mathcal{CN}(\mu, \sigma^2)$  denotes a complex circularly symmetric Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$ . Similarly,  $\mathcal{N}_m(\mu, \Sigma)$  denotes a complex  $m$ -variate Gaussian distribution with a mean vector  $\mu \in \mathbb{C}^m$  and a covariance matrix  $\Sigma \in \mathbb{C}^{m \times m}$ .

$$\mathbb{P}_{\text{Opp-AF}}(\text{outage}) = \mathbb{P} \left\{ \frac{1}{2} \log_2 \left( 1 + \max_{k \in \mathcal{S}_{\text{relay}}} \underbrace{\frac{\gamma_{Sk} \gamma_{kD}}{\frac{N_0}{\mathcal{P}_R} (1 + \eta_{Sk}) + \gamma_{kD}}}_{\triangleq W_k^{(\text{AF})}} \frac{\mathcal{P}_S}{N_0} \right) < R \right\} \quad (3)$$

path between the source and destination) is given by [5], [6]

$$\mathcal{I}_{\text{MR-AF}} = \frac{1}{2} \log_2 \left\{ 1 + \frac{\left| \sum_{k=1}^K \sqrt{\frac{v_k \mathcal{P}_R}{\Omega_{Sk} \mathcal{P}_S + N_0}} \alpha_{Sk} \alpha_{kD} \right|^2 \mathcal{P}_S}{\left( 1 + \sum_{k=1}^K \frac{v_k \mathcal{P}_R |\alpha_{kD}|^2}{\Omega_{Sk} \mathcal{P}_S + N_0} \right) N_0} \right\}. \quad (1)$$

For any given power allocation  $(\mathcal{P}_S, \mathcal{P}_R)$ , we are looking for the optimal set  $\{v_k\}_{k=1}^K$  (subject to  $\sum_{k=1}^K v_k = 1$  and  $0 \leq v_k \leq 1$ ) that minimizes the outage probability:

$$\begin{aligned} & \text{minimize} && \mathbb{P} \{ \mathcal{I}_{\text{MR-AF}} < R \} \\ & \text{subject to} && \sum_{k=1}^K v_k = 1 \\ & && 0 \leq v_k \leq 1, \quad \forall k \in \mathcal{S}_{\text{relay}}. \end{aligned} \quad (2)$$

Note that with *single* opportunistic relaying, we should select a relay that maximizes the argument of the logarithm in (1), yielding the outage probability (3), shown at the top of the page. This minimizes the outage probability among all *single*-relay selection rules in AF relay environments.<sup>2</sup>

#### IV. OUTAGE OPTIMALITY

In the following theorem, we show that the opportunistic single-relay selection in (3) outperforms any other strategy that employs simultaneous, in-band, AF transmissions from more than one relay under the aggregate power constraint.

*Theorem 1:* For AF relaying with the aggregate power constraint  $\mathcal{P}_R$ , choosing the “best” relay

$$b_{\text{AF}}^* = \arg \max_{k \in \mathcal{S}_{\text{relay}}} W_k^{(\text{AF})} \quad (4)$$

is globally outage-optimal, that is,

$$\begin{aligned} \mathbb{P}_{\text{MR-AF}}(\text{outage}) &= \mathbb{P} \{ \mathcal{I}_{\text{MR-AF}} < R \} \\ &\geq \mathbb{P}_{\text{Opp-AF}}(\text{outage}). \end{aligned} \quad (5)$$

*Proof:* Let

$$\begin{aligned} \mathbf{h}_1 &= [h_{1,1} \quad h_{1,2} \quad \dots \quad h_{1,K}]^T \sim \tilde{\mathcal{N}}_K(\mathbf{0}, \mathbf{I}_K) \\ \mathbf{h}_2 &= [h_{2,1} \quad h_{2,2} \quad \dots \quad h_{2,K}]^T \sim \tilde{\mathcal{N}}_K(\mathbf{0}, \mathbf{I}_K) \end{aligned}$$

be independent complex  $K$ -dimensional (column) Gaussian vectors, where the superscript  $(\cdot)^T$  denotes the transpose and  $\mathbf{I}_K$  is the  $K \times K$  identity matrix. Then, defining the  $K \times K$  diagonal matrices

$$\mathbf{G}_1 = \text{diag}(\sqrt{\eta_{S1}}, \sqrt{\eta_{S2}}, \dots, \sqrt{\eta_{SK}}) \quad (6)$$

$$\mathbf{G}_2 = \text{diag} \left( \sqrt{\frac{\eta_{1D}}{\eta_{S1} + 1}}, \sqrt{\frac{\eta_{2D}}{\eta_{S2} + 1}}, \dots, \sqrt{\frac{\eta_{KD}}{\eta_{SK} + 1}} \right) \quad (7)$$

and

$$\mathbf{V} = \text{diag}(\sqrt{v_1}, \sqrt{v_2}, \dots, \sqrt{v_K}), \quad (8)$$

we can rewrite (1) as

$$\begin{aligned} \mathcal{I}_{\text{MR-AF}} &= \frac{1}{2} \log_2 \left\{ 1 + \frac{|\mathbf{h}_2^\dagger \mathbf{G}_1 \mathbf{V} \mathbf{G}_2 \mathbf{h}_1|^2}{1 + \mathbf{h}_2^\dagger (\mathbf{V} \mathbf{G}_2)^2 \mathbf{h}_2} \right\} \\ &= \frac{1}{2} \log_2 \left\{ 1 + \mathbf{h}_1^\dagger \boldsymbol{\Sigma}_{(\mathbf{h}_2)} \mathbf{h}_1 \right\} \end{aligned} \quad (9)$$

where  $\dagger$  denotes the transpose conjugate and  $\boldsymbol{\Sigma}_{(\mathbf{h}_2)} \in \mathbb{C}^{K \times K}$  is given by

$$\boldsymbol{\Sigma}_{(\mathbf{h}_2)} = \frac{\mathbf{G}_2 \mathbf{V} \mathbf{G}_1 \mathbf{h}_2 \mathbf{h}_2^\dagger \mathbf{G}_1 \mathbf{V} \mathbf{G}_2}{1 + \mathbf{h}_2^\dagger (\mathbf{V} \mathbf{G}_2)^2 \mathbf{h}_2}. \quad (10)$$

Let  $\lambda_{[1]} \geq \lambda_{[2]} \geq \dots \geq \lambda_{[K]}$  be the ordered eigenvalues of  $\boldsymbol{\Sigma}_{(\mathbf{h}_2)}$ . Then, since  $\boldsymbol{\Sigma}_{(\mathbf{h}_2)}$  is a positive semidefinite matrix of one rank, we have  $\lambda_{[2]} = \lambda_{[3]} = \dots = \lambda_{[K]} = 0$  and nonzero positive eigenvalue  $\lambda_{[1]}$  equal to

$$\lambda_{[1]} = \frac{\mathbf{h}_2^\dagger (\mathbf{G}_1 \mathbf{V} \mathbf{G}_2)^2 \mathbf{h}_2}{1 + \mathbf{h}_2^\dagger (\mathbf{V} \mathbf{G}_2)^2 \mathbf{h}_2}. \quad (11)$$

Since  $\boldsymbol{\Sigma}_{(\mathbf{h}_2)}$  is Hermitian, there exists a unitary matrix  $\mathbf{U} \in \mathcal{U}(K)$  such that  $\boldsymbol{\Sigma}_{(\mathbf{h}_2)} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^\dagger$  and  $\boldsymbol{\Lambda} = \text{diag}(\lambda_{[1]}, 0, 0, \dots, 0)$ , where  $\mathcal{U}(K)$  denotes a unitary group of  $K \times K$  unitary matrices, i.e.,

$$\mathcal{U}(K) = \left\{ \mathbf{U} \in \mathbb{C}^{K \times K} : \mathbf{U} \mathbf{U}^\dagger = \mathbf{I}_K \right\}.$$

<sup>2</sup>The analytical result for  $\mathbb{P}_{\text{Opp-AF}}(\text{outage})$  can be found in [5].

Therefore, we have

$$\begin{aligned}
\mathbb{P}_{\text{MR-AF}}(\text{outage}) &= \mathbb{P}\left\{\mathbf{h}_1^\dagger \Sigma(\mathbf{h}_2) \mathbf{h}_1 < 2^{2R} - 1\right\} \\
&\stackrel{(a)}{=} \mathbb{P}\left\{\mathbf{h}_1^\dagger \Lambda \mathbf{h}_1 < 2^{2R} - 1\right\} \\
&= \mathbb{P}\left\{|h_{1,1}|^2 \lambda_{[1]} < 2^{2R} - 1\right\} \\
&\stackrel{(b)}{\geq} \mathbb{P}\left\{|h_{1,1}|^2 \max_{k \in \mathcal{S}_{\text{relay}}} \left(\frac{\frac{\eta_{S_k} \eta_{kD}}{\eta_{S_k} + 1} |h_{2,k}|^2}{1 + \frac{\eta_{kD}}{\eta_{S_k} + 1} |h_{2,k}|^2}\right) < 2^{2R} - 1\right\} \\
&\geq \mathbb{P}\left\{\max_{k \in \mathcal{S}_{\text{relay}}} \left(\frac{\frac{\eta_{S_k} \eta_{kD}}{\eta_{S_k} + 1} |h_{1,k}|^2 |h_{2,k}|^2}{1 + \frac{\eta_{kD}}{\eta_{S_k} + 1} |h_{2,k}|^2}\right) < 2^{2R} - 1\right\} \\
&= \mathbb{P}\left\{\max_{k \in \mathcal{S}_{\text{relay}}} W_k^{(\text{AF})} < \frac{2^{2R} - 1}{\frac{P_s}{N_0}}\right\} \\
&\stackrel{(c)}{=} \mathbb{P}_{\text{Opp-AF}}(\text{outage}) \tag{12}
\end{aligned}$$

where (a) follows from the fact that the distribution of the complex Gaussian vector  $\mathbf{h}_1$  is unitary invariant, i.e.,

$$\mathbf{U}^\dagger \mathbf{h}_1 \sim \tilde{\mathcal{N}}_K(\mathbf{0}, \mathbf{I}_K), \quad \forall \mathbf{U} \in \mathcal{U}(K) \tag{13}$$

while (b) and (c) follow from Lemma 1 in Appendix and (3), respectively.  $\square$

## V. DISCUSSION

Theorem 1 demonstrates that cooperation is useful even when AF relays choose not to transmit. In fact, this outage optimality reveals that optimal relay selection minimizes the outage probability and outperforms any other strategy that involves simultaneous transmissions from more than one AF relay.

We note that in-band transmissions from multiple relays require a) some type of non-trivial space-time coding, and b) RF-front ends that support simultaneous reception of multiple in-band signals and thus, depart from the conventional low-cost radios treating additional in-band signals as noise. In contrast, selection and single relay retransmission allow the use of existing low-cost radios. The main difficulty in relay selection is the discovery of the optimal relay in a timely fashion with small overhead. Such distributed techniques were proposed and analyzed in [4] and tested with low-cost radios in [6].

Intuitively, the outage optimality of opportunistic AF relaying comes at no surprise. Simultaneous transmissions from more than one AF relays do not necessarily add constructively at the receiver, since each relay transmission arrives at the destination with different phase (the channel gain  $\alpha_{AB}$  between any two points is a complex number). Therefore, the sum is not necessarily greater than the individual transmissions and it is preferable to utilize a single AF relay transmission. Alternatively, the relays could adjust their phase before retransmission in such ways that their signals add in-phase at the destination.<sup>3</sup>

<sup>3</sup>This is also true for Gaussian relay channels where the channel gain is considered as a real number (no phase).

This is the main idea behind distributed phased-arrays, which a) require complex radio hardware and b) radically deviate from the basic motivation of AF: simplicity of intermediate relays. This work demonstrates that cooperation benefits can be maximized without the need of distributed phased-arrays or any complex radio hardware, but instead, with intelligent scheduling among low-cost AF relays.

## APPENDIX

*Lemma 1 (Bound of Ratio of Scaled Positive Numbers):*

For  $a_k > 0$ ,  $b_k > 0$ , and  $c_k \geq 0$ ,  $k = 1, 2, \dots, K$ , with  $\sum_{k=1}^K c_k = 1$ , the following inequality holds:

$$\frac{\sum_{k=1}^K a_k b_k c_k}{1 + \sum_{k=1}^K b_k c_k} \leq \max_{k \in \{1, 2, \dots, K\}} \frac{a_k b_k}{1 + b_k}. \tag{14}$$

*Proof:* Let

$$k^* = \arg \max_{k \in \{1, 2, \dots, K\}} \frac{a_k b_k}{1 + b_k}. \tag{15}$$

Then, substituting  $c_{k^*} = 1 - \sum_{j \neq k^*} c_j$  and taking into account the fact that all involved terms are nonnegative, the inequality (14) is equivalent to

$$\sum_{j \neq k^*} \left[ a_j b_j - a_{k^*} b_{k^*} + b_{k^*} b_j (a_j - a_{k^*}) \right] c_j \leq 0. \tag{16}$$

The inequality (16) always holds since

$$a_j b_j - a_{k^*} b_{k^*} + b_{k^*} b_j (a_j - a_{k^*}) \leq 0, \quad \forall j \neq k^* \tag{17}$$

yielding the desired result (14).  $\square$

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